Stabilization of Stochastic Quantum Dynamics via Open and Closed Loop Control

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Introduction

- Stochastic Master Equation (SME) for quantum filtering
- Averaging over noise: Markovian Master Equation (MME)
- Invariance & attractivity of subspaces for MME
  $\Longleftrightarrow$ Global Asymptotic Stability (GAS)
- Environment-assisted stabilization:
  - block-structure of the dissipation/measurement operators
  - open-loop control
- Invariance & attractivity of subspaces for the SME
  $\Longleftrightarrow$ Global Asymptotic Stability in probability
- Same environment-assisted stabilization properties for MME and SME
- Feedback-assisted stabilization for SME
- Examples

Quantum filtering

- quantum filtering: Stochastic Master Equation (SME) à la Itô:
  - SME

$$d\rho_t = \left(\mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t) \right) dt + \mathcal{G}(M, \rho_t) dW_t$$

  - $\rho_t \in \mathfrak{D}(\mathcal{H}) =$ set of density operators in n-dimensional Hilbert space $\mathcal{H}$
  - continuous weak measurement: output equation

$$dY_t = \sqrt{\eta}\frac{1}{2}\text{tr}(\rho_t(M + M^\dagger))dt + dW_t$$

    - $M =$ measurement operator $\in \mathfrak{B}(\mathcal{H})$
    - $\eta \in [0, 1] =$ efficiency of the measurement
    - dW_t = "innovation process"
  - example: homodyne detection

Stochastic Master Equation (SME)

$$d\rho_t = \left(\mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t) \right) dt + \mathcal{G}(M, \rho_t) dW_t$$

- Hamiltonian

$$\mathcal{F}(H, \rho) := -i[H, \rho]$$

- Lindbladian dissipation

$$\mathcal{D}(L_k, \rho) := L_k \rho L_k^\dagger - \frac{1}{2}(L_k^\dagger L_k \rho + \rho L_k^\dagger L_k)$$

- measurement
  - drift

$$\mathcal{D}(M, \rho) := M \rho M^\dagger - \frac{1}{2}(M^\dagger M \rho + \rho M^\dagger M)$$

  - diffusion

$$\mathcal{G}(M, \rho) := \sqrt{\eta}(M\rho + \rho M^\dagger - \text{tr}((M + M^\dagger)\rho)\rho)$$

Stochastic Master Equation (SME)

- infinitesimal generator

$$\mathcal{A}[\cdot] = \frac{1}{2}\text{tr}\Bigg(\Big(\mathcal{F}(H,\rho_t) + \sum_k \mathcal{D}(L_k,\rho_t) + \mathcal{D}(M,\rho_t)\Big)\frac{\partial\,[\cdot]}{\partial\rho} + \frac{\partial\,[\cdot]}{\partial\rho}\Big(\mathcal{F}(H,\rho_t) + \sum_k \mathcal{D}(L_k,\rho_t) + \mathcal{D}(M,\rho_t)\Big) + \frac{1}{2}\mathcal{G}^2(M,\rho_t))\frac{\partial^2\,[\cdot]}{\partial\rho^2} + \frac{1}{2}\frac{\partial^2\,[\cdot]}{\partial\rho^2}\mathcal{G}^2(M,\rho_t))\Bigg)$$

- solution:

$$\rho_t = \mathcal{T}_t^W(\rho_0), \quad \rho_0 \in \mathfrak{D}(\mathcal{H})$$

$$= \rho_0 + \int_0^t \left(\mathcal{F}(H,\rho_s) + \sum_{k=1}^r \mathcal{D}(L_k,\rho) + \mathcal{D}(M,\rho_s)\right) ds + \int_0^t \mathcal{G}(M,\rho_s)dW_s$$

  - ρ_t $\exists$ uniquely
  - ρ_t adapted to the filtration $\mathcal{E}_t$ associated to $\{W_t, t\in\mathbb{R}^+\}$
  - ρ_t $\mathfrak{D}(\mathcal{H})$-invariant by construction

Markovian Master Equation (MME)

- averaging the SME over the noise trajectories
- $\Longrightarrow$Markovian Master Equation

$$\dot{\rho}(t) = \mathcal{L}(\rho(t)) = \mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t)$$

- Quantum dynamical semigroup $\mathcal{T}(\rho)$ is a TPCP (Trace-Preserving Completely Positive) map

$$\rho(t) = \mathcal{T}_t(\rho_0), \quad \rho_0 \in \mathfrak{D}(\mathcal{H})$$

$$= \exp \int_0^t \left(\mathcal{F}(H, \rho_s) + \sum_{k=1}^r \mathcal{D}(L_k, \rho) + \mathcal{D}(M, \rho_s) \right) ds$$

- Assumption: Hamiltonian H can be chosen arbitrarily

State space decomposition

F. Ticozzi, L. Viola. Quantum Markovian subsystems: Invariance, attractivity and control, *IEEE Tr. Autom. Contr.*, 2008

- State decomposition

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$$

  - $\mathcal{H}_S$ = "target" subspace, $\dim(\mathcal{H}_S) = s$
  - $\mathcal{H}_R$ = "remainer" subspace, $\dim(\mathcal{H}_R) = n - s$
  - $\Longrightarrow$block structure for densities in $\mathfrak{D}(\mathcal{H})$

$$\rho = \left(\begin{array}{c|c} \rho_S & \rho_P \\ \hline \rho_Q & \rho_R \end{array}\right)$$

  - $\Longrightarrow$block structure for operators on $\mathcal{H}$

$$H = \left(\begin{array}{c|c} H_S & H_P \\ \hline H_P^\dagger & H_R \end{array}\right), \quad L = \left(\begin{array}{c|c} L_S & L_P \\ \hline L_Q & L_R \end{array}\right), \quad M = \left(\begin{array}{c|c} M_S & M_P \\ \hline M_Q & M_R \end{array}\right)$$

Invariance & attractivity of $\mathcal{H}_S$ for the MME

$$\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$$

- density initialized in $\mathcal{H}_S$

$$\mathfrak{I}_S(\mathcal{H}) = \left\{ \rho \in \mathfrak{D}(\mathcal{H}) \,|\, \rho = \begin{bmatrix} \rho_S & 0 \\ 0 & 0 \end{bmatrix}, \ \rho_S \in \mathfrak{D}(\mathcal{H}_S) \right\}$$

Definition: *$\mathcal{H}_S$ is an invariant subspace for the system under the TPCP maps $\{\mathcal{T}_t(\cdot)\}_{t\geq 0}$ if $\mathfrak{I}_S(\mathcal{H})$ is an invariant subset of $\mathfrak{D}(\mathcal{H})$, i.e. if*

$$\rho \in \mathfrak{I}_S(\mathcal{H}) \quad \Longrightarrow \quad \mathcal{T}_t(\rho) \in \mathfrak{I}_S(\mathcal{H}) \quad \forall\, t \geq 0$$

Definition: *$\mathcal{H}_S$ supports an attractive subsystem with respect to a family of TPCP maps $\{\mathcal{T}_t\}_{t\geq 0}$ if $\forall \rho \in \mathfrak{D}(\mathcal{H})$ the following condition is asymptotically obeyed*

$$\lim_{t\to\infty} \left\| \mathcal{T}_t(\rho) - \left(\begin{array}{c|c} \rho_S(t) & 0 \\ \hline 0 & 0 \end{array} \right) \right\| = 0.$$

- $\mathcal{H}_S$ invariant and attractive $\Longleftrightarrow$ Globally Asymptotically Stable (GAS)

Invariance for the MME

Theorem *$\mathcal{H}_S$ supports an invariant subspace iff*

$$L_k = \left(\begin{array}{c|c} L_{S,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array}\right) \ \forall\ k, \qquad M = \left(\begin{array}{c|c} M_S & M_P \\ \hline 0 & M_R \end{array}\right)$$

$$iH_P - \frac{1}{2}\left(\sum_k L_{S,k}^\dagger L_{P,k} + M_S^\dagger M_P\right) = 0.$$

- idea of the proof:

$$\frac{d}{dt}\rho = \left(\begin{array}{c|c} \mathcal{L}_S(\rho) & \mathcal{L}_P(\rho) \\ \hline \mathcal{L}_Q(\rho) & \mathcal{L}_R(\rho) \end{array}\right) \qquad \forall\ \rho_S \in \mathfrak{I}_S(\mathcal{H}) \Longrightarrow \left(\begin{array}{c|c} \mathcal{L}_S(\rho_S) & 0 \\ \hline 0 & 0 \end{array}\right)$$

- necessary and sufficient conditions on the structure of the blocks of H, L_k and M
  - $M_Q = 0$ and $L_{Q,k} = 0\ \forall k$
  - Hamiltonian H_P is used to compensate for $L_{P,k} \neq 0$ and/or $M_P \neq 0 \Longrightarrow$open-loop "matching"

Attractivity for the MME

Theorem *Assume $\mathcal{H}_S$ supports an invariant subsystem. Then $\mathfrak{I}_S(\mathcal{H})$ can be made attractive iff $\mathfrak{I}_R(\mathcal{H})$ is not invariant.*

- idea: $L_{P,k} \neq 0$ and/or $M_P \neq 0 \Longrightarrow \mathcal{H}_R$ can never be made invariant by Hamiltonian compensation
- if $L_{P,k} = L_{Q,k} = 0$ and $M_P = M_Q = 0$ then $\mathfrak{I}_S(\mathcal{H})$ and $\mathfrak{I}_R(\mathcal{H})$ both invariant (when $H_P = 0$) or non-invariant ($H_P \neq 0$)
- if $L_{P,k} = L_{Q,k} \neq 0$ and/or $M_P = M_Q \neq 0$ then neither $\mathfrak{I}_S(\mathcal{H})$ nor $\mathfrak{I}_R(\mathcal{H})$ can be invariant

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- if $L_{P,k} = L_{Q,k} \neq 0$ and/or $M_P = M_Q \neq 0$ then neither $\mathfrak{I}_S(\mathcal{H})$ nor $\mathfrak{I}_R(\mathcal{H})$ can be invariant
- **Summary:** rendering $\mathcal{H}_S$ attractive for the MME is “to the expenses of $\mathcal{H}_R$”, and can be accomplished by means of
  1. non-hermitian L_k and/or non-hermitian M $\Longrightarrow$(block) “ladder-like” operators
  2. open-loop control

  - $\mathcal{H}_S$ invariant and attractive $\Longleftrightarrow$ Globally Asymptotically Stable (GAS)

$\Longrightarrow$environment-assisted stabilization

Extension to SME

- Problem formulation: global asymptotic stability for the SME
- $\mathfrak{I}_S(\mathcal{H})$ globally asymptotically stable in probability $\iff \mathfrak{I}_S(\mathcal{H})$ is invariant and attractive in probability a.s.

Problem *Consider the SME*

$$d\rho_t = \left(\mathcal{F}(H, \rho_t) + \sum_k \mathcal{D}(L_k, \rho_t) + \mathcal{D}(M, \rho_t) \right) dt + \mathcal{G}(M, \rho_t) dW_t$$

and a target subspace $\mathcal{H}_S$ such that $\mathcal{H} = \mathcal{H}_S \oplus \mathcal{H}_R$.
Do there exist

- *dissipation operators L_k and/or measurement operator M*
- *Hamiltonian H*

such that the target set $\mathfrak{I}_S(\mathcal{H})$ is invariant under $\mathcal{T}_t^W(\cdot)$ and attractive in probability a.s.?

Condition for invariance

$$d\rho = \left(\begin{array}{c|c} \mathcal{L}_S(\rho) & \mathcal{L}_P(\rho) \\ \hline \mathcal{L}_Q(\rho) & \mathcal{L}_R(\rho) \end{array}\right) dt + \left(\begin{array}{c|c} \mathcal{G}_S(M,\rho) & \mathcal{G}_P(M,\rho) \\ \hline \mathcal{G}_Q(M,\rho) & \mathcal{G}_R(M,\rho) \end{array}\right) dW_t$$

- Approach: in order for $\mathfrak{I}_S(\mathcal{H})$ to be invariant for the SME, it has to be invariant for both its diffusion and drift parts

$$\forall\, \rho_S \in \mathfrak{I}_S(\mathcal{H}) \quad \Longrightarrow \quad \left(\begin{array}{c|c} \mathcal{L}_S(\rho_S) & 0 \\ \hline 0 & 0 \end{array}\right), \quad \left(\begin{array}{c|c} \mathcal{G}_S(M,\rho_S) & 0 \\ \hline 0 & 0 \end{array}\right)$$

$\Longrightarrow$ block structure of L_k and M matrices

- for $\rho_S \in \mathfrak{I}_S(\mathcal{H})$
  - diffusion

$$\mathcal{G}(M,\rho_S) = \left(\begin{array}{c|c} (M_S - \mathrm{Tr}(M_S\rho_S))\rho_S + \rho_S(M_S^\dagger - \mathrm{Tr}(M_S^\dagger\rho_S)) & \rho_S M_Q^\dagger \\ \hline M_Q\rho_S & 0 \end{array}\right)$$

$\Longrightarrow \mathfrak{I}_S(\mathcal{H})$ is invariant to $\mathcal{G}(M,\rho_S)$ if $M_Q = 0$

Condition for invariance

- drift

$$\mathcal{F}(H,\rho_S)+\sum_k \mathcal{D}(L_k,\rho_S)+\mathcal{D}(M,\rho_S)=$$

$$\left[\begin{array}{c|c} -iH_S\rho_S+i\rho_S H_S+d_1(L_k,M,\rho_S) & i\rho_S H_P+\sum_k d_2(L_k,\rho_S)-\frac{1}{2}\rho_S M_S^\dagger M_P \\ \hline * & \sum_k L_{Q,k}\rho_S L_{Q,k}^\dagger \end{array}\right]$$

where we define:

$$d_1(L_k,M,\rho_S)=M_S\rho_S M_S^\dagger-\frac{1}{2}(M_S^\dagger M_S\rho_S+\rho_S M_S^\dagger M_S)+\sum_k \Big\{L_{S,k}\rho_S L_{S,k}^\dagger$$

$$-\frac{1}{2}(L_{S,k}^\dagger L_{S,k}\rho_S+\rho_S L_{S,k}^\dagger L_{S,k}+L_{Q,k}^\dagger L_{Q,k}\rho_S+\rho_S L_{Q,k}^\dagger L_{Q,k})\Big\},$$

$$d_2(L_k,\rho_S)=L_{S,k}\rho_S L_{Q,k}^\dagger-\frac{1}{2}\rho_S(L_{S,k}^\dagger L_{P,k}+L_{Q,k}^\dagger L_{R,k}).$$

- $\Longrightarrow$same conditions on H_P, M_P and $L_{P,k}$ as for the MME

Invariance for the SME

- putting together

Proposition $\mathfrak{I}_S(\mathcal{H})$ *is invariant for the SME iff*

$$L_k = \left(\begin{array}{c|c} L_{S,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array}\right) \quad \forall k,$$

$$M = \left(\begin{array}{c|c} M_S & M_P \\ \hline 0 & M_R \end{array}\right),$$

$$iH_P - \frac{1}{2}\left(\sum_k L_{S,k}^\dagger L_{P,k} + M_S^\dagger M_P\right) = 0,$$

Corollary
$\mathfrak{I}_S(\mathcal{H})$ *invariant for the SME* $\iff$ $\mathfrak{I}_S(\mathcal{H})$ *invariant for the MME*

- More rigorous proof: support theorem

Attractivity for the SME

- attractivity of SME $\Longleftrightarrow$ attractivity of MME

Proposition *Assume* $\mathfrak{I}_S(\mathcal{H})$ *is attractive for the MME for some* H, L_k *and* M. *Then with these* H, L_k *and* M, $\mathfrak{I}_S(\mathcal{H})$ *attractive in probability also for the SME*

Attractivity for the SME

- attractivity of SME $\Longleftrightarrow$ attractivity of MME

> **Proposition** *Assume $\mathfrak{I}_S(\mathcal{H})$ is attractive for the MME for some H, L_k and M. Then with these H, L_k and M, $\mathfrak{I}_S(\mathcal{H})$ attractive in probability also for the SME*

- **Proof:**

  1. stability of $\mathfrak{I}_S(\mathcal{H})$ in probability
     - candidate Lyapunov function

$$V(\rho) = \text{tr}(\Pi_R \rho)$$

where $\Pi_R =$ projection on $\mathcal{H}_R$

$$V(\rho) = 0 \text{ for } \rho \in \mathfrak{I}_S(\mathcal{H})$$

$$V(\rho) > 0 \text{ for } \rho \notin \mathfrak{I}_S(\mathcal{H})$$

$$\mathcal{A}V(\rho) = -\text{tr}\left(\left(\sum_k L_{P,k}^\dagger L_{P,k} + M_P^\dagger M_P\right)\rho_R\right) \leqslant 0 \quad \forall \rho \in \mathfrak{D}(\mathcal{H})$$

by ciclicity of the trace

Attractivity for the SME (cont.)

2. attractivity (by contradiction)

- Suppose $\exists\ \rho_0 \in \mathfrak{D}(\mathcal{H}) \setminus \mathfrak{I}_S(\mathcal{H})$ for which $\mathfrak{I}_S(\mathcal{H})$ is not attractive

$$\Longrightarrow \quad P\left(\lim_{t\to\infty} \mathfrak{d}(\mathcal{T}_t^W(\rho_0), \mathfrak{I}_S(\mathcal{H})) = 0\right) = 1 - p, \quad p > 0$$

$$\Longrightarrow \quad P\left(\lim_{t\to\infty} V(\mathcal{T}_t^W(\rho_0)) = 0\right) = 1 - p$$

$\Longrightarrow$ for $t \to \infty$ set $\{V(\mathcal{T}_t^W(\rho_0)) > 0\}$ has measure > 0

- in expectation then $\exists\ \xi(\rho_0) > 0$ for which

$$\limsup_{t\to\infty} E\left[V(\mathcal{T}_t^W(\rho_0))\right] = \limsup_{t\to\infty} \int_{\{V(\mathcal{T}_t^W(\rho_0))>0\}} V(\mathcal{T}_t^W(\rho_0)) dP$$

$$\geq \xi(\rho_0) \limsup_{t\to\infty} \int_{\{V(\mathcal{T}_t^W(\rho_0))>0\}} dP$$

$$= \xi(\rho_0) \limsup_{t\to\infty} \left\{1 - P\left(V(\mathcal{T}_t^W(\rho_0)) = 0\right)\right\} = \xi(\rho_0) p$$

$\Longrightarrow \exists k > 0$ s.t. $\quad \limsup_{t\to\infty} \mathfrak{d}(E[\mathcal{T}_t^W(\rho_0)], \mathfrak{I}_S(\mathcal{H})) \geq \dfrac{\xi(\rho_0)p}{k} > 0.$

Invariance & attractivity of SME

- **Summary:** Invariance and attractivity of MME
  $\Longleftrightarrow$ invariance and attractivity of the SME
  $\Longleftrightarrow$ GAS of MME and SME
- both can be accomplished by means of
  1. non-hermitian L_k and/or non-hermitian M
     $\Longrightarrow$(block) "ladder-like" operators
  2. open-loop control

Theorem $\mathfrak{I}_S(\mathcal{H})$ *is GAS in probability for the SME* $\Longleftrightarrow$ L_k, M *and* H *have the structure*

$$L_k = \left(\begin{array}{c|c} L_{S,k} & L_{P,k} \\ \hline 0 & L_{R,k} \end{array}\right) \quad \forall k, \quad M = \left(\begin{array}{c|c} M_S & M_P \\ \hline 0 & M_R \end{array}\right),$$

$$iH_P - \frac{1}{2}\left(\sum_k L_{S,k}^\dagger L_{P,k} + M_S^\dagger M_P\right) = 0,$$

with at least one $L_{P,k} \neq 0$ *and/or* $M_P \neq 0$

$\Longrightarrow$environment-assisted stabilization

Block diagonal case

- how about the case

$$L_k = \left(\begin{array}{c|c} L_{S,k} & 0 \\ \hline 0 & L_{R,k} \end{array}\right) \quad \forall k, \qquad M = \left(\begin{array}{c|c} M_S & 0 \\ \hline 0 & M_R \end{array}\right)$$

- open-loop, time-invariant control: both $\mathcal{H}_S$ and $\mathcal{H}_R$ are invariant $\Longrightarrow$neither is attractive

Proposition *Given SME with L_k and M as above, no time-invariant H exists rendering $\mathfrak{I}_S(\mathcal{H})$ GAS in probability*

- **Proof:** if $H_P = 0$ then $\mathfrak{I}_R(\mathcal{H})$ is invariant;
  if instead $H_P \neq 0$ then $\mathfrak{I}_S(\mathcal{H})$ cannot be invariant

Feedback-assisted stabilization

- **Assumptions:** $\dim(\mathcal{H}_S) = 1$ (pure state stabilization)
  $\dim(\mathcal{H}) \geqslant 3$
- further split of $\mathcal{H}_R$: $\mathcal{H} = \mathcal{H}_S \oplus \underbrace{\mathcal{H}_C \oplus \mathcal{H}_Z}_{\mathcal{H}_R}$

$$H = H_c + u(\rho) H_f = \left(\begin{array}{c|c|c} 0 & 0 & 0 \\ \hline 0 & H_{c,C} & H_{c,W} \\ \hline 0 & H_{c,W}^\dagger & H_{c,Z} \end{array}\right) + u(\rho) \left(\begin{array}{c|c|c} 0 & H_{f,U} & 0 \\ \hline H_{f,U}^\dagger & 0 & 0 \\ \hline 0 & 0 & 0 \end{array}\right)$$

$$L_k = \left(\begin{array}{c|c|c} L_{k,S} & 0 & 0 \\ \hline 0 & L_{k,C} & L_{k,W} \\ \hline 0 & L_{k,Y} & L_{k,Z} \end{array}\right) \qquad M = \left(\begin{array}{c|c|c} M_S & 0 & 0 \\ \hline 0 & M_C & M_W \\ \hline 0 & M_Y & MZ \end{array}\right)$$

- idea:
  - design $H_{c,W}$ so as to keep "reshuffling" inside $\mathcal{H}_R = \mathcal{H}_C \oplus \mathcal{H}_Z$
  - use $H_{f,U}$ to "drain" probability out of $\mathcal{H}_R$

Feedback-assisted stabilization

- use "patchy" feedback law similar to

  M. Mirrahimi, R. van Handel. Stabilizing feedback controls for quantum systems, SIAM J. Control Optim., **46**, 445-467, 2007

Theorem *Given SME with* L_k *and* M *as above and* $H = H_c + u(\rho_t)H_f$, *where the feedback control law* $u(\rho_t)$ *s.t. for* $\rho_d \in \mathcal{H}_S$

1. *If* $\text{tr}(\rho_t\rho_d) \geq \gamma \;\; u(\rho_t) = -\text{tr}(i[H_f, \rho_t]\rho_d)$
2. *If* $\text{tr}(\rho_t\rho_d) \leq \gamma/2, \;\; u(\rho_t) = 1$;
3. *If* $\rho_t \in \mathcal{B} = \{\rho : \gamma/2 < \text{tr}(\rho_t\rho_d) < \gamma\}$, *then*
   - $u(\rho_t) = -\text{tr}(i[H_f, \rho_t]\rho_d)$ *if* ρ_t *last entered* $\mathcal{B}$ *through the boundary* $\text{tr}(\rho_d\rho) = \gamma$,
   - $u_t = 1$ *otherwise.*

Then $\exists\, \gamma > 0$ *such that* $u(\rho_t)$ *renders the SME GAS in probability*

- differences with M. Mirrahimi, R. van Handel:
  - valid for more general class of Lindbladians
  - uses feedback in a "minimal" way (to enable state transitions otherwise impossible)
  - uses the environment as much as possible

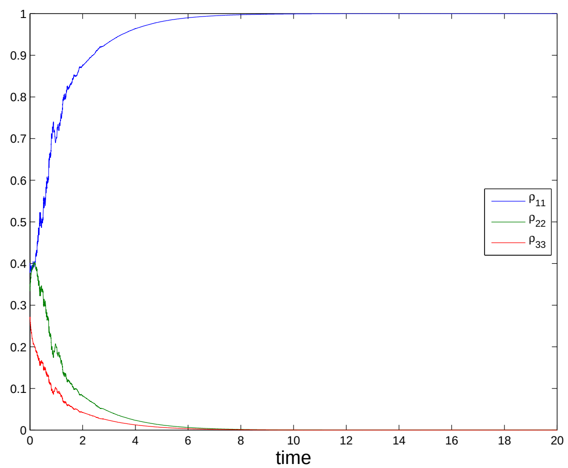
Example 1: environment only

$$\rho_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 3 \\ 0 & 2 & 1 \end{bmatrix}$$

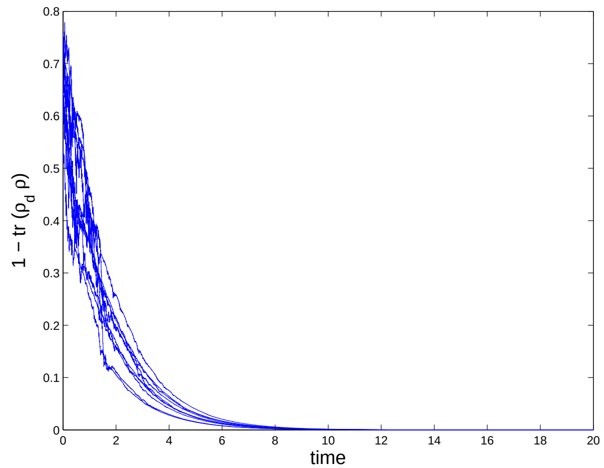
- $\mathcal{H}_S$ is GAS without any open/closed loop Hamiltonian

$$H_c = 0 \qquad H_f = 0$$

energy population



sample trajectories



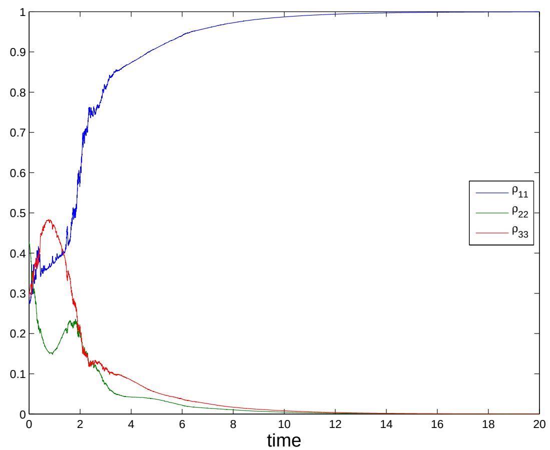
Example 2: environment + open loop

$$\rho_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad L_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

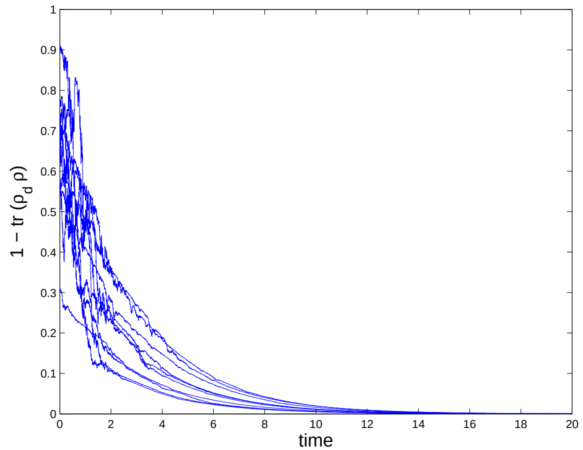
- $\mathcal{H}_S$ is rendered GAS by the following open loop Hamiltonian

$$H_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad H_f = 0$$

energy population



sample trajectories



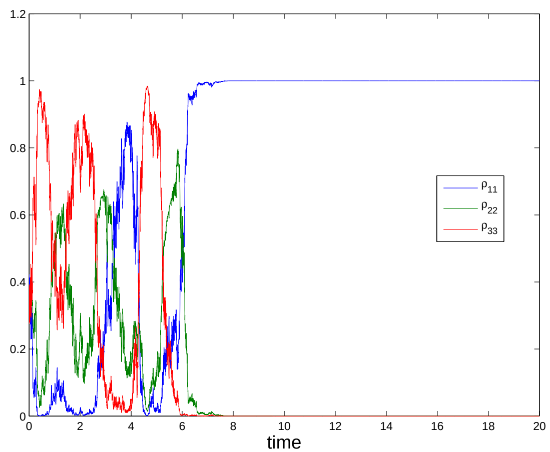
Example 3: open loop + closed loop

$$\rho_d = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \qquad L_1 = 0$$

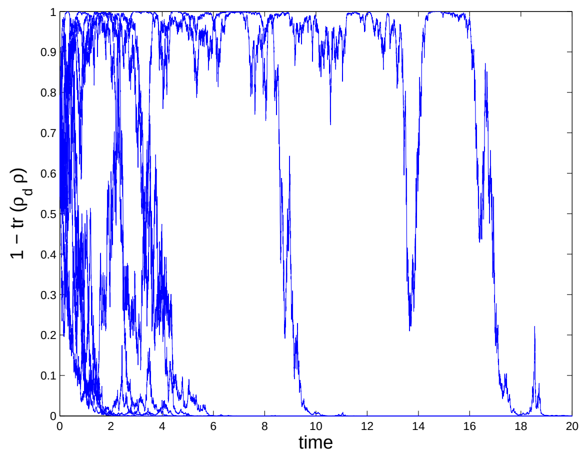
- $\mathcal{H}_S$ is rendered GAS by the open loop and feedback Hamiltonians

$$H_c = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \qquad H_f = \begin{bmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

energy population



sample trajectories



Conclusion

- Environment (i.e., dissipation) and open-loop control can easily lead to conditions for GAS of a subsystem in both MME and SME
- Philosophy: make the most use of environment and the least of feedback
- Crucial ingredients:
  - non-hermitian part (”ladder operator”) in the dissipation and/or measurement operator
  - open-loop control design by ”matching”
- GAS of MME $\Longleftrightarrow$ GAS of SME $\Longleftrightarrow$ invariance & attractivity
- When environment and open loop are not enough: feedback can be useful to get GAS

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