Control and stabilization of the Bloch equation

Karine Beauchard (CNRS, CMLS, Ecole Polytechnique)

joint works with

Jean-Michel Coron (LJLL, Paris 6)
Pierre Rouchon (CAS, Mines de Paris)
Paulo Sergio Pereira da Silva (Escola Politecnica Sao Polo)

QUAINT, Dijon, April, 9, 2013

Plan

The Bloch equation

An ensemble of non interacting spins, in a magnetic field $B(t) := (u(t), v(t), B_0)$, with dispersion in the Larmor frequency $\omega = \gamma B_0 \in (\omega_*, \omega^*)$ (=rotation speed around z).

$$\text{one spin :} \quad M(t,\omega) \in S^2$$

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \;\; \omega \in (\omega_*, \omega^*)$$

State : M **Controls :** u, v

controllability of an ODE, simultaneously w.r.t. $\omega \in (\omega_*, \omega^*)$

Li-Khaneja(06)

Applications : Nuclear Magnetic Resonance

Controllability question for the Bloch equation

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in [0,+\infty) \times (\omega_*, \omega^*)$$

Ex : $M_0(\omega) \equiv -e_3, \quad M_f(\omega) \equiv +e_3$,
But spins with different ω have different dynamics !

Goal : Use the control to compensate for the dispersion in ω.

Rk : If ω is fixed, the controllability of one ODE on S^2 is trivial.

Goals of this talk :

- show that the pioneer works by Li and Khaneja can be completed by functionnal analysis methods to go further,
- propose different approaches relying on Fourier analysis or feedback stabilization.

A prototype for infinite dimensional bilinear systems with continuous spectrum

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad \omega \in (\omega_*, \omega^*)$$

$$\mathcal{A}M := \omega e_3 \wedge M(\omega) \qquad \rightarrow \qquad \mathrm{Sp}(\mathcal{A}) = -i(\omega_*, \omega^*) \bigcup i(\omega_*, \omega^*)$$

$$\lambda = \pm i\widetilde{\omega} \qquad \rightarrow \qquad M_\lambda(\omega) = \begin{pmatrix} 1 \\ \mp i \\ 0 \end{pmatrix} \delta_{\widetilde{\omega}}(\omega)$$

$\Rightarrow$ Toy model for $\quad i\partial_t \psi = (-\Delta + V)\psi - u(t)\mu(x)\psi$

State of the art : bilinear control for Schrödinger PDEs

Quite well understood :
exact controllability 1D

- negative results : Ball-Marsden-Slemrod(82), Turinici(00), Ilner-Lange-Teismann(06), Mirrahimi-Rouchon(04) Nersesyan(10).
- positive local results with discrete spectrum + gap (1D) : KB(05), KB-Laurent(09, 11,), Morancey(12), KB-Lange-Teismann(12)....
- positive global results : KB-Coron(06), Nersesyan(09).

approximate controllability with discrete spectrum
Chambrion-Mason-Sigalotti-Boscain-Boussaïd-Caponigro(09,...), Nersesyan(09), Ervedoza-Puel(09).

Not well understood : with continuous spectrum : Mirrahimi(09)

1st part

Control of the linearized system around $(M \equiv e_3, u \equiv v \equiv 0)$

Linearized system around $(M \equiv e_3, u \equiv v \equiv 0)$

$$\begin{cases} \partial_t M(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad \omega \in (\omega_*, \omega^*) \\ M(0,\omega) = e_3 \end{cases}$$

If $\quad (u,v) = \epsilon(\widetilde{u},\widetilde{v}) \quad$ then $\quad M = e_3 + \epsilon \widetilde{M} + \underset{\epsilon \to 0}{o}(\epsilon) \quad$ where

$$\begin{cases} \partial_t \widetilde{M}(t,\omega) = \omega e_3 \wedge \widetilde{M} + \Big[u(t)e_1 + v(t)e_2\Big] \wedge e_3 \\ \widetilde{M}(0,\omega) = 0 \end{cases}$$

The notations

$$\widetilde{M} := (x,y,z), \qquad \mathcal{Z}(t,\omega) := (x+iy)(t,\omega), \qquad w(t) := (v - iu)(t)$$

lead to the very simple ODE

$$\frac{d\mathcal{Z}}{dt}(t,\omega) = i\omega\mathcal{Z}(t,\omega) + w(t)$$

Linearized system around ($M \equiv e_3, u \equiv v \equiv 0$) : non exact controllability, approximate controllability

$$\mathcal{Z}(T,\omega) = \left(\mathcal{Z}_0(\omega) + \int_0^T w(t) e^{-i\omega t} dt \right) e^{i\omega T}$$

- $T > 0$, the reachable set from $\mathcal{Z}_0 = 0$ is $\mathcal{F}[L^1(-T,0)]$
- the $\mathcal{Z}_0$ asymptotically zero controllable are $\mathcal{F}[L^1(0,+\infty)]$
- $\forall \mathcal{Z}_0$ in that space, the control is unique
- $\forall T > 0$, approximate controllability in $C^0[\omega_*, \omega^*]$ with $C_c^\infty(0,T)$-controls.

We will see that the NL syst has better controllability properties.

2nd part

Nonlinear system : no exact controllability with a priori bounded controls

Whole space : structure of the reachable set

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in (0,T) \times \mathbb{R}$$

Theorem : Let $T > 0$ and $R := 1/(8\sqrt{3T})$.

- $\forall u, v \in B_R[L^2(0,T)], \exists ! M = (x,y,z)$ solution with $\mathcal{Z} := x + iy \in C^0([0,T], L^2(\mathbb{R})) \cap C^0_b([0,T] \times \mathbb{R})$,
- the image of

$$\begin{array}{rccc} F_T : & B_R[L^2(0,T)]^2 & \to & L^2 \cap C^0_b(\mathbb{R}) \\ & (u,v) & \mapsto & \mathcal{Z}(T,.) \end{array}$$

is a non flat **submanifold** of $L^2 \cap C^0_b(\mathbb{R})$, with ∞ codim.

Proof : Inverse mapping $dF_T(0,0).(U,V) \sim \mathcal{F}(U+iV) + 2^{nd}$ order

On a bounded interval : analyticity argument

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in (0,T) \times (\omega_*, \omega^*)$$

- $T > 0, u, v \in L^2(0,T) \Rightarrow \mathcal{Z}(T,.)$ analytic
- $T > 0$, $R := 1/(8\sqrt{3T})$.
  There exists arbitrarily small **analytic** targets that cannot be reached exactly in time T with controls in $B_R[L^2(0,T)]$.

The non controllability is not a question of regularity.

3rd part

Nonlinear system : approximate controllability with unbounded controls

Solutions associated to Dirac controls

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in (0,T) \times (\omega_*, \omega^*)$$

Classical solution for $u, v \in L^1_{loc}(\mathbb{R})$.

If $u = \alpha\delta_a$ and $v = 0$ then

$$M(a^+,\omega) = \exp(\alpha\Omega_x) M(a^-,\omega)$$

$\rightarrow$ instantaneous rotation of angle α around the x-axis, $\forall\omega$

Rk : limit $[\epsilon \rightarrow 0]$ of solutions associated to $u = \frac{\alpha}{\epsilon} 1_{[a,a+\epsilon]}$.

Approximate controllability result $-\infty < \omega_* < \omega^* < +\infty$

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in [0,+\infty) \times (\omega_*, \omega^*)$$

Theorem : Let $M_0 \in H^1((\omega_*,\omega^*), S^2)$. There exist $(t_n)_{n\in\mathbb{N}} \in [0,+\infty)^{\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ finite sums of Dirac masses such that

$$U[t_n^+; u_n, v_n, M_0] \to e_3 \text{ weakly in } H^1.$$

Rk : Same result with $u, v \in L^\infty_{loc}[0,+\infty)$: $\quad \alpha\delta_a \leftarrow \frac{\alpha}{\epsilon} 1_{[a,a+\epsilon]}$
Approximate controllability in H^s, $\forall s < 1$, in L^∞...

First step : Li-Khaneja 's non commutativity result

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad (t,\omega) \in [0,+\infty) \times (\omega_*, \omega^*)$$

Theorem : Let $P, Q \in \mathbb{R}[X]$. $\forall \epsilon > 0, \exists \tau^* > 0$ such that
$\forall \tau \in (0,\tau^*)$, $\exists T > 0, u, v \sim$ Dirac such that

$$\Big\| U[T^+; u, v, .] - \Big(I + \tau[P(\omega)\Omega_x + Q(\omega)\Omega_y] \Big) \Big\|_{H^1(\omega_*,\omega^*)} \leqslant \epsilon\tau.$$

Proof : Explicit controls $\rightarrow$ cancel the drift term, Lie brackets.

Rk : It is not sufficient for the global approximate controllability.
$\tau\omega^N$ needs $T_N \sim 2^N \tau^{\frac{1}{N}}$ and more than 2^N N-S.

Second step : variationnal argument

Let $M_0 \in H^1((\omega_*, \omega^*), S^2)$ be such that $M_0 \neq e_3$.

Goal : Find $U[t_n^+; u_n, v_n, M_0] \rightharpoonup e_3$ in H^1 when $n \to +\infty$

$$K := \left\{ \widetilde{M} \,;\, \exists U[t_n^+; u_n, v_n, M_0] \rightharpoonup \widetilde{M} \text{ in } H^1, [...] \right\}$$

$$m := \inf \left\{ \|\widetilde{M}'\|_{L^2} ; \widetilde{M} \in K \right\}$$

1) $\exists e \in K$ such that $m = \|e'\|_{L^2}$
2) $m = 0$. Otherwise, one may decrease more : $\exists P, Q \in \mathbb{R}[X]$ st

$$\left\| \frac{d}{d\omega} \left[\Big(I + \tau [P(\omega)\Omega_x + Q(\omega)\Omega_y] \Big) e \right] \right\|_{L^2} < \|e'\|_{L^2}$$

3) $e_3 \in K \cap S^2$

Conclusion

Theorem : Let $M_0 \in H^1((\omega_*, \omega^*), S^2)$. There exist $(t_n)_{n\in\mathbb{N}} \in [0, +\infty)^{\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}, (v_n)_{n\in\mathbb{N}}$ finite sums of Dirac masses such that

$$U[t_n^+; u_n, v_n, M_0] \to e_3 \text{ weakly in } H^1.$$

Advantages :

- global result
- strong convergence in H^s, $\forall s < 1$, L^∞

Flaws : How to do ? The strategy of the proof may

- not work,
- take a long time,
- cost a lot (N-S).

4th part

Nonlinear system : explicit controls for the asymptotic exact controllability

Explicit controls for the asymptotic exact controllability

Notations : - $(\omega_*, \omega^*) = (0, \pi)$, $f : (0, \pi) \to \mathbb{C}$ identified with $\tilde{f} : \mathbb{R} \to \mathbb{R}$, 2π periodic symmetric, $N(f) := \sum_{n \in \mathbb{Z}} |c_n(f)|$.

- $M = (x, y, z)$, $\mathcal{Z} := x + iy$

Theorem : $\exists \delta > 0$ / $\forall M_0 : (0, \pi) \to S^2$ with $N[\mathcal{Z}_0] < \delta$ and $z_0 > 1/2$, the solution of the Bloch equation with

$$u(t) := \pi \delta_k(t) - \sum_{p=1}^{2k-1} \Im\Big(c_{-k+p}(\mathcal{Z}_0)\Big)\delta_{k+p}(t) + \pi\delta_{3k}(t),$$

$$v(t) := -\sum_{p=1}^{2k-1} \Re\Big(c_{-k+p}(\mathcal{Z}_0)\Big)\delta_{k+p}(t),$$

where $k = k(\mathcal{Z}_0)$ / $\sum_{|n|>k} |c_n(\mathcal{Z}_0)| < N(\mathcal{Z}_0)/4$ satisfies

$$N[\mathcal{Z}(3k^+)] < \frac{N(\mathcal{Z}_0)}{2} \quad \text{and} \quad z(3k^+) > 1/2.$$

Ideas of the proof

1) 'cancel' $c_n(\mathcal{Z}_0)$ **for** $n \leqslant 0$ with $w(t) = \sum_{k=0}^{N} c_{-k}\delta_k(t)$

$$\mathcal{Z}(N^+,\omega) \sim \left(\mathcal{Z}_0(\omega) - \int_0^N w(t)e^{-i\omega t}dt\right)e^{i\omega N}$$

$$\sim \left(\sum_{n\in\mathbb{Z}} c_n e^{in\omega} - \sum_{k=0}^{N} c_{-k}e^{-ik\omega}\right)e^{i\omega N}$$

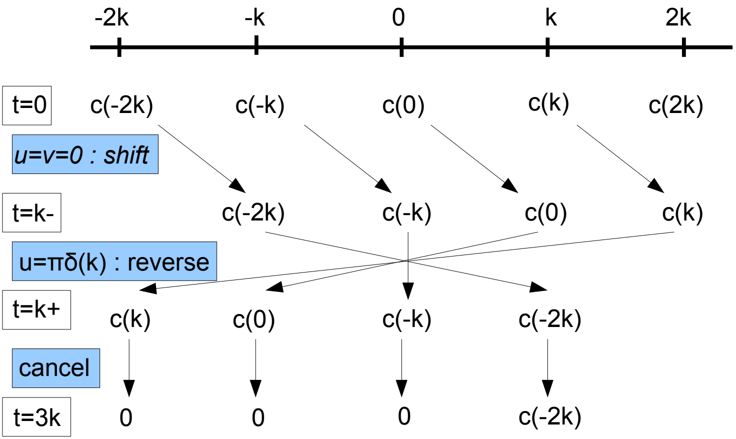
2) shift to the right with $u \equiv v \equiv 0$,

$$\mathcal{Z}(N,\omega) = \mathcal{Z}_0(\omega)e^{iN\omega} = \sum_{n\in\mathbb{Z}} c_n e^{i(n+N)\omega}$$

3) reverse with $u(t) = \pi\delta_0(t)$, $M(0^+) = \exp(\pi\Omega_x)M_0$

$$\mathcal{Z}(0^+,\omega) = \overline{\mathcal{Z}_0(\omega)} = \sum_{n\in\mathbb{Z}} \overline{c_n}e^{-in\omega}$$

Proof


-2k
-k
0
k
2k
t=0
c(-2k)
c(-k)
c(0)
c(k)
c(2k)
u=v=0 : shift
t=k-
c(-2k)
c(-k)
c(0)
c(k)
u=πδ(k) : reverse
t=k+
c(k)
c(0)
c(-k)
c(-2k)
cancel
t=3k
0
0
0
c(-2k)

5th part

Nonlinear system : feedback stabilization

Goal

$$\partial_t M(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \omega e_3\Big] \wedge M(t,\omega), \quad \omega \in (\omega_*, \omega^*)$$

Propose **explicit feedback laws** $u = u(M)$, $v = v(M)$ that stabilize the Bloch equation around a uniform state of spin $+1/2$ or $-1/2$.

$$M(t,\omega) \underset{t\to+\infty}{\longrightarrow} e_3 \quad \text{uniformly wrt} \quad \omega \in (\omega_*, \omega^*)$$

Interest : less sensible to random perturbations than open loop controls

Strategy

Feedback design tool : control Lyapunov function

Convergence for ODEs : LaSalle invariance principle

Convergence for PDEs : several adaptions
- approximate stabilization : with discrete [KB-Mirrahimi(09)] or continuous spectrum [Mirrahimi(09)]
- weak stabilization : under a strong compactness assumption [Ball-Slemrod(79)] or without [here, KB-Nersesyan(10), Morancey (12)]
- strong stabilization : with compact trajectories [d'Andréa-Novel-Coron(98)] or strict Lyapunov functions [Coron-d'Andréa-Novel-Bastin(07)]

The impulse train structure control

In view of the previous results, it is natural to consider

$$u = u_{smooth} + \sum_{k=1}^{\infty} \pi\delta(t - kT)$$

$$(x, y, z)(kT^+) = (x, -y, -z)(kT^-)$$

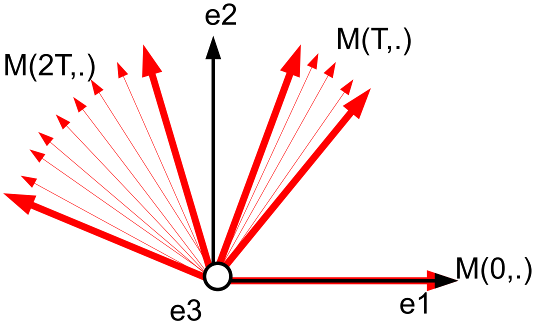
With $\epsilon(t) = (-1)^{E(t/T)}$, the change of variables

$$(x, y, z) \leftarrow (x, \epsilon(t)y, \epsilon(t)z), \qquad u \leftarrow u + \sum_{k=1}^{\infty} \pi\delta(t - kT), \qquad v \leftarrow \epsilon(t)v$$

transforms the Bloch equation into

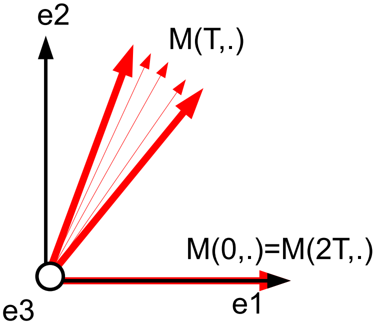
$$\frac{\partial M}{\partial t}(t, \omega) = \Big[u(t)e_1 + v(t)e_2 + \epsilon(t)\omega e_3\Big] \wedge M(t, \omega)$$

The impulse train structure reduces the dispersion



Initial free system

The impulse train structure reduces the dispersion



New free system

Driftless form

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)e_1 + v(t)e_2 + \epsilon(t)\omega e_3\Big] \wedge M(t,\omega)$$

$$M(t,\omega) \longleftarrow \mathcal{R}_z\left(-\omega \int_0^t \epsilon(s)ds\right) M(t,\omega)$$

$$\frac{\partial M}{\partial t}(t,\omega) = \Big[u(t)W_1(t,\omega) + v(t)W_2(t,\omega)\Big] \wedge M(t,\omega)$$

Interest : Any fonction may be a (controlled) Lyapunov function.

Control design : control Lyapunov function

$$\mathcal{L}(t) := \int_{\omega_*}^{\omega^*} \left[\left\| \frac{\partial M}{\partial \omega}(t,\omega) \right\|^2 + z(t,\omega) \right] d\omega$$

$$\frac{d\mathcal{L}}{dt}(t) = \Re\left[\Omega(t)\mathcal{H}(t)\right]$$

where

$$\mathcal{H}(t) := \int_{\omega_*}^{\omega^*} \left[i\zeta(t)[\overline{\mathcal{Z}}z' - \overline{\mathcal{Z}}' z] - \overline{\mathcal{Z}(t,\omega)} \right] e^{-i\omega\zeta(t)} d\omega$$

So we take

$$\Omega(t) := -\overline{\mathcal{H}(t)} \qquad \text{then} \qquad \frac{d\mathcal{L}}{dt}(t) = -|\Omega(t)|^2$$

Local stabilization

Theorem : There exists $\delta > 0$ such that, for every $M_0 \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with $\|M_0 + e_3\|_{H^1} < \delta$, the solution of the closed loop system satisfies

$$M(t) \rightharpoonup -e_3 \text{ in } H^1(\omega_*, \omega^*) \text{ when } t \to +\infty.$$

Rk : $M(t, \omega) \to -e_3$ uniformly with respect to $\omega \in (\omega_*, \omega^*)$.

Proof : 1. Invariant set $= \{-e_3\}$ locally.
2. $\Omega(t) \to 0$ a.e.
3. $-e_3$ is the only possible weak H^1-limit :
If $M(t_n) \to M_\infty^0$ weakly in H^1 and strongly in $H^{1/2}$ then $M(t_n + \tau) \to M_\infty(\tau)$ strongly in $H^{1/2}$, $\forall \tau > 0$, thus $\Omega[M(t_n + \tau)] \to \Omega[M_\infty(\tau)]$. Therefore $\Omega[M_\infty] \equiv 0$.

Key point : $\Omega(M)$ is well defined for M only in $H^{1/2}$

No global stabilization

Topological obstructions : $H^1((\omega_*, \omega^*), S^2)$ cannot be continuously deformed to one point.

Actually, there is an infinite number of invariant solutions, that may be expressed explicitly...

Numerical simulations

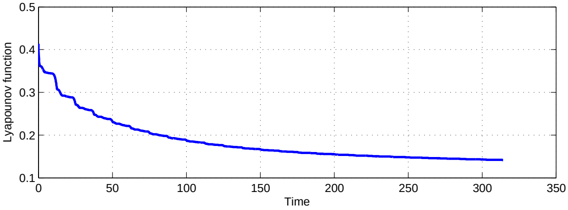
Parameters : $(\omega_*, \omega^*) = (0, 1)$, $T = 2\pi$, $G := 1/(2T^2)$

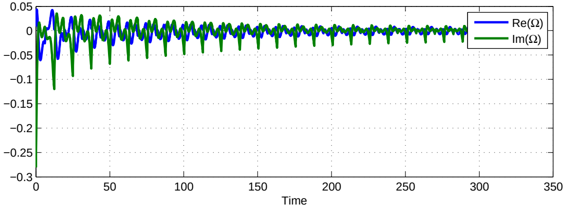
$$\begin{pmatrix} x_0(\omega) \\ y_0(\omega) \\ z_0(\omega) \end{pmatrix} := \begin{pmatrix} \cos(\pi, \omega)\sqrt{1 - z_0(\omega)^2} \\ \sin(\pi, \omega)\sqrt{1 - z_0(\omega)^2} \\ 0.8 - 0.1\sin(4\pi\omega) \end{pmatrix}.$$

Simulation until $T_f = 50T$

Conclusion : The convergence speed is rapid at the beginning but decreases at the end.

Numerical simulations





Adaptation to stabilize a variable profile

Target : $M_f(\omega)$

- Find $R(\omega) \in SO_3$ such that $R(\omega)M_f(\omega) = -e_3$.
- Force $N(t,\omega) := R(\omega)M(t,\omega) \to -e_3$ when $t \to +\infty$ (same strategy).

Proof OK for

- $M_f \in H^1((\omega_*, \omega^*), \mathbb{S}^2)$ with $\langle M_f(\omega), e_3 \rangle \neq 0, \forall \omega \in (\omega_*, \omega^*)$,
- $\|M_0 - M_f\|_{H^1}$ small enough.

Numerical simulations

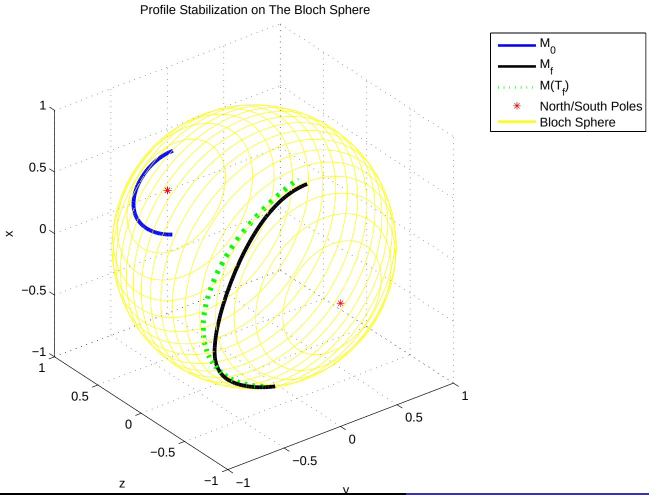
Parameters : $\omega_* = 0$, $\omega^* = 1$, $T = 2\pi$

$$\begin{pmatrix} x_0(\omega) \\ y_0(\omega) \\ z_0(\omega) \end{pmatrix} := \begin{pmatrix} -0.35\sin(0.9\pi\omega - \pi/2) \\ -0.35\cos(\pi\omega - \pi/2) \\ \sqrt{1 - x_0(\omega)^2 - y_0(\omega)^2} \end{pmatrix}$$

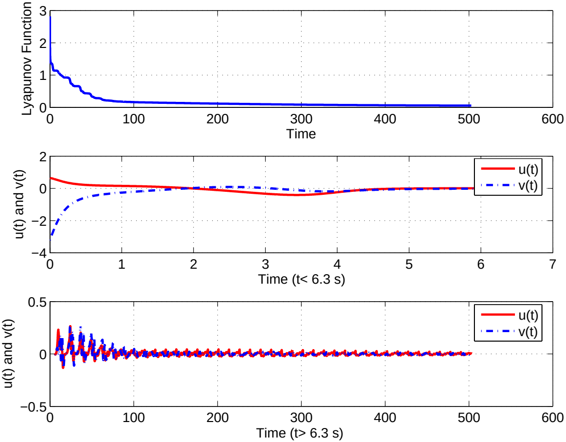
$$\begin{pmatrix} x_f(\omega) \\ y_f(\omega) \\ z_f(\omega) \end{pmatrix} := \begin{pmatrix} -0.79\sin(\pi\omega - \pi/2) \\ -0.79\cos(0.9\pi\omega - \pi/2) \\ -\sqrt{1 - x_f(\omega)^2 - y_f(\omega)^2} \end{pmatrix}$$

Simulation until $T_f = 80T$

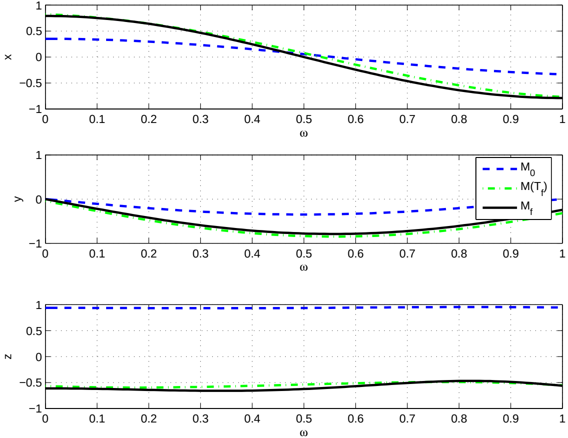
Numerical simulations



Numerical simulations



Numerical simulations



Conclusion of the talk : Controllability

Linearized system :

- non exact controllability, L^1 controls : $\mathcal{F}[L^1(-T,0)]$
- non asymptotic zero controllability
- uniqueness of the control
- approximate controllability, unbounded controls

Nonlinear system :

- non exact controllability, $B_R[L^2(0,T)]$-controls : manifold
- approximate controllability in H^s, $s < 1$, unbounded controls : *non commutativity + variationnal method*
- explicit controls for the (local) asymptotic exact controllability to e_3 : *Fourier method,* many controls work

The nonlinearity allows to recover controllability.

Conclusion of the talk : Stabilization

- impulse train control
- driftless form
- control Lyapunov function : H^1-distance to the target
- explicit damping feedback laws
- weak H^1 local stabilization

Open problems, perspectives

- **exact** controllability in **finite time** with unbounded controls ?
- **strong** stabilization with the same feedback laws ?
- explicit feedbacks for the **semi-global** stabilization
- convergence rates ? arbitrarily fast stabilization ?

References

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