Week 2 workshop exercises – limits

1. Guess the explicit and the recurrent expressions for the following sequences:
   (a) 1, 4, 7, 10, 13, ...  (b) 1, 3, 9, 27, 81, ...  (c) 1, –1/5, 1/25, –1/125, 1/625, ...

2. Find the first six terms of the following recursively defined sequences:
   (a) \( u_{n+1} = u_n + n, \quad u_0 = 1 \)  (b) \( u_{n+2} = 2u_{n+1} + u_n, \quad u_0 = 1, \quad u_1 = 1 \)

3. Find the limits for \( r \to 0 \) and \( r \to \infty \) of the following functions:
   (a) \( f(r) = \frac{1}{3r} \)  (b) \( f(r) = 2^r \)  (c) \( f(r) = \frac{1}{r+2} \)
   (d) \( f(r) = \frac{r}{r+2} \)  (e) \( f(r) = \frac{r}{r^2 + r + 1} \)  (f) \( f(r) = \frac{3r^2 + 3r + 1}{5r^2 - 6r - 1} \)

4. Find the limits for \( x \to 0 \) and \( x \to \infty \) of the following functions:
   (a) \( f(x) = x^2 e^{-x} \)  (b) \( f(x) = \cos(2x) e^{-x} \)

5*. Einstein’s model for the molar heat capacity \( C_V \) of a crystalline solid held at constant volume yields the following expression:

\[
C_V(T) = 3R \left( \frac{h\nu}{kT} \right)^2 \frac{e^{h\nu/kT}}{(e^{h\nu/kT} - 1)^2}
\]

where \( \nu \) is the frequency of crystal lattice vibrations. Using a substitution \( x = h\nu/kT \), find the zero-temperature limit of this expression.

6*. The radial wavefunction for the 3s atomic orbital of the hydrogen atom has the following form:

\[
\psi(r) = N \left( \frac{r}{a_0} \right)^2 \exp \left( -\frac{r}{a_0} \right)
\]

where \( N \) is the normalisation constant and \( a_0 \) is Bohr’s radius. Find the limits of this function for \( r \to 0 \) and \( r \to \infty \), and explain the physical meaning of your results.

7. Show that the function \( y = x^2 \) is continuous for any value of the argument \( x \). Use the definition: demonstrate that this function is equal to its left and right limit at every point.

8. Prove that, if the function \( f(x) \) is continuous and non-negative in the interval \((a,b)\), then the function \( F(x) = \sqrt{f(x)} \) is also continuous in that interval. Proceed as follows: show that the square root is a continuous function, and then the superposition property.
9. For which values of $x$ is the function $f(x) = \tan x$ discontinuous and why?

10. A function is defined by the formula:

$$f(x) = \begin{cases} \frac{x^2 - 4}{x - 2} & \text{for } x \neq 2 \\ a & \text{for } x = 2 \end{cases}$$

how should one choose the value of $a = f(2)$ to heal the “puncture” at $x = 2$ and make the function continuous at that point?

11. By inventing a suitable example, demonstrate that the sum of two discontinuous functions may be a continuous function.

12*. Find points of discontinuity and singularity (if any) for the following functions:

(a) $y = \frac{x^2}{x - 2}$

(b) $y = \frac{1 + x^3}{1 + x}$

(c) $y = \frac{x}{|x|}$

(d) $y = \exp \left[ \frac{1}{x + 1} \right]$