Week 2 workshop exercises – limits

1. Guess the explicit and the recurrent expressions for the following sequences:

(a) 1, 4, 7, 10, 13, … (b) 1, 3, 9, 27, 81, … (c) 1, −1/5, 1/25, −1/125, 1/625, …

2. Find the first six terms of the following recursively defined sequences:

(a) $u_{n+1} = u_n + n, \quad u_0 = 1$ (b) $u_{n+2} = 2u_{n+1} + u_n, \quad u_0 = 1, \quad u_1 = 1$

3. Find the limits for $r \to 0$ and $r \to \infty$ of the following functions:

(a) $f(r) = \dfrac{1}{3^r}$ (b) $f(r) = 2^r$ (c) $f(r) = \dfrac{1}{r+2}$

(d) $f(r) = \dfrac{r}{r+2}$ (e) $f(r) = \dfrac{r}{r^2+r+1}$ (f) $f(r) = \dfrac{3r^2+3r+1}{5r^2-6r-1}$

4. Find the limits for $x \to 0$ and $x \to \infty$ of the following functions:

(a) $f(x) = x^2 e^{-x}$ (b) $f(x) = \cos(2x)e^{-x}$

5*. Einstein's model for the molar heat capacity C_{V} of a crystalline solid held at constant volume yields the following expression:

$$C_{\mathrm{V}}(T) = 3R\left(\frac{h\nu}{kT}\right)^2 \frac{e^{h\nu/kT}}{\left(e^{h\nu/kT}-1\right)^2}$$

where ν is the frequency of crystal lattice vibrations. Using a substitution $x = h\nu/kT$, find the zero-temperature limit of this expression.

6*. The radial wavefunction for the 3s atomic orbital of the hydrogen atom has the following form:

$$\psi(r) = N\left(\frac{r}{a_0}\right)^2 \exp\left(-\frac{r}{a_0}\right)$$

where N is the normalisation constant and a_0 is Bohr's radius. Find the limits of this function for $r \to 0$ and $r \to \infty$, and explain the physical meaning of your results.

7. Show that the function $y = x^2$ is continuous for any value of the argument x. Use the definition: demonstrate that this function is equal to its left and right limit at every point.

8. Prove that, if the function $f(x)$ is continuous and non-negative in the interval (a,b), then the function $F(x) = \sqrt{f(x)}$ is also continuous in that interval. Proceed as follows: show that the square root is a continuous function, and then the superposition property.

9. For which values of x is the function $f(x) = \tan x$ discontinuous and why?

10. A function is defined by the formula:

$$f(x) = \begin{cases} \dfrac{x^2 - 4}{x - 2} & \text{for } x \neq 2 \\ a & \text{for } x = 2 \end{cases}$$

how should one choose the value of $a = f(2)$ to heal the "puncture" at $x = 2$ and make the function continuous at that point?

11. By inventing a suitable example, demonstrate that the sum of two discontinuous functions may be a continuous function.

12*. Find points of discontinuity and singularity (if any) for the following functions:

(a) $y = \dfrac{x^2}{x-2}$ (b) $y = \dfrac{1+x^3}{1+x}$ (c) $y = \dfrac{x}{|x|}$ (d) $y = \exp\left[\dfrac{1}{x+1}\right]$