Week 6 workshop exercises

1. Find general solutions for the following differential equations:

   (a) \( \frac{dy}{dx} = \frac{3x^2}{y} \)
   
   (b) \( \frac{dy}{dx} = 4xy^2 \)
   
   (c) \( \frac{dx}{dt} = 2(x+1)t \)

2. Solve the initial value problems for the following ODEs:

   (a) \( \frac{dy}{dx} = \frac{y+1}{x-3}, \quad y(0) = 1 \)
   
   (b) \( \frac{dy}{dx} = \frac{x^2-1}{2y+1}, \quad y(0) = -1 \)

   Note: in part (b), the quadratic equation for \( y \) may be left unsolved.

3. Find the interval \( \tau_{1/2} \) in which the amount of reactant \( A \) in a first order decay process

   \[ A(t) = A(0)e^{-kt} \]

   is reduced by a factor of 2.

4. The following function was proposed as an approximation to \( \cos(x) \) around \( x = 0 \):

   \[ f(x) = 1 - \frac{115x^2}{252} + \frac{313x^4}{15120} + \frac{11x^2}{252} + \frac{13x^4}{15120} \]

   Using a calculator, inspect the performance of this approximation (a) at the origin; (b) in the interval between \(-\pi/2\) and \(+\pi/2\); (c) around \( x = 2\pi \); (d) at infinity. The easiest way to proceed is to pick a grid of \( x \) points, and to compare the values of the function and the approximation at those points.

5. Using the substitution \( x = kT/\Delta E \), demonstrate (by calculating the function for a few values of \( x \), or by carefully plotting both functions) that the following expression

   \[ Q(T) = \frac{1}{2} + \frac{kT}{\Delta E} + \frac{\Delta E}{12kT} \]

   can be an excellent approximation to the partition function of the ensemble of harmonic oscillators

   \[ Q(T) = \frac{1}{1 - e^{-\Delta E/kT}} \]

   Use your calculator to estimate the interval of \( kT/\Delta E \) for which the approximation is good (you would first need to decide what is to be considered “good”).

6. Show that a differential equation of the form

   \( \frac{dy}{dx} = f(ax + by + c) \)

   where \( a, b, c \) are constants, is reduced to a separable form by the substitution \( u = ax + by \). Proceed by calculating the differential \( du \) and using the result to eliminate \( dy \) from the numerator.