Week 7 workshop exercises

1. Calculate the first six terms of the Taylor series around \( x_0 = 0 \) for the following functions:

   \[
   \begin{align*}
   (a) \quad & \frac{1}{1 - 3x} \\
   (b) \quad & \frac{1}{1 + 5x^2} \text{ (think first)} \\
   (c) \quad & \frac{1}{2 + x}
   \end{align*}
   \]

   and use the ratio test to find the values of \( x \) for which the series converge.

2. Find the linear high-temperature approximation for the Fermi-Dirac distribution that describes the average number of fermions \( n \) in a single-particle state with the energy \( U \) in a system with the total chemical potential \( \mu \):

   \[n(T) = \frac{1}{e^{(U-\mu)/kT} + 1}\]

   Note: make a substitution \( x = (U - \mu)/kT \), get the Taylor series up to the linear term around \( x = 0 \), then reverse the substitution.

3. Find the radius of convergence for each of the following series:

   \[
   \begin{align*}
   (a) \quad & \sum_{m=0}^{\infty} \frac{x^m}{4^m} \\
   (b) \quad & \sum_{r=0}^{\infty} (-1)^r x^{2r} \\
   (c) \quad & \sum_{n=1}^{\infty} nx^n \\
   (d) \quad & \sum_{n=1}^{\infty} \frac{x^n}{n^2}
   \end{align*}
   \]

4. The following are the Taylor series for sine and cosine:

   \[
   \begin{align*}
   \sin x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \\
   \cos x &= \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}
   \end{align*}
   \]

   (a) write out the first four terms of each series explicitly;

   (b) differentiate the explicit expression for the sine series and demonstrate that the derivative is equal to the cosine series;

   (b) integrate the explicit expression for the sine series and demonstrate that the integral is equal to the negative of the cosine series plus a constant.

5. Find the Lagrange interpolant for the function taking the following values: \( f(0) = 0, f(1) = 1, f(2) = 1, f(3) = 2 \). Calculate the value of the interpolant at \( x = 3/2 \).