

$$\textcircled{1} \quad \text{a) } \vec{x} = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 2 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 3 \end{pmatrix}$$

$$|\vec{x}| = \sqrt{1+16+0+4} = \sqrt{21}$$

$$|\vec{y}| = \sqrt{4+4+1+9} = \sqrt{18}$$

$$\langle \vec{x} | \vec{y} \rangle = 1 \cdot 2 - 4 \cdot 2 + 0 \cdot 1 + 2 \cdot 3 = 0$$

$$\text{b) } \vec{x} = \begin{pmatrix} 0 \\ i \\ 0 \\ -i \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 1 \end{pmatrix}$$

$$|\vec{x}| = \sqrt{0+1+0+1} = \sqrt{2}$$

$$|\vec{y}| = \sqrt{0+1+1+1} = \sqrt{3}$$

$$\begin{aligned} \langle \vec{x} | \vec{y} \rangle &= \sum_k x_k^* y_k = \\ &= 0 \cdot 0 + (-i) \cdot 1 + 0 \cdot 1 - (-i) \cdot 1 = \\ &= -i + i = 0. \end{aligned}$$

$$\text{c) } \vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad \vec{y} = \begin{pmatrix} 0 \\ i \\ 0 \\ i \end{pmatrix} \quad |\vec{x}| = \sqrt{0+1+0+1} = \sqrt{2}$$

$$|\vec{y}| = \sqrt{0+1+0+1} = \sqrt{2}$$

$$\langle \vec{x} | \vec{y} \rangle = 1 \cdot 0 + 0 \cdot i + 1 \cdot 0 + 0 \cdot i = 0.$$

$\textcircled{2}$ No, because the basis of a two-dimensional space has two vectors. Any other vector is by definition their linear combination. Therefore any set of three 2D vectors is linearly dependent.

③ All properties are satisfied. Because any 3×3 matrix has 9 independent elements, the space is 9-dimensional. Other possibilities are functions (infinite-dimensional), differential operators (infinite-dimensional), and bank accounts (one-dimensional).

④ None of the four vectors is a linear combination of other three.

⑤ By direct inspection:

$$\begin{pmatrix} 3 \\ 7i \\ 8 \\ 0 \end{pmatrix} = 3\vec{x}_1 + 7i\vec{x}_2 + 8\vec{x}_3.$$

$$\textcircled{6} \quad \vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \vec{x}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\vec{y} = \begin{pmatrix} i \\ 0 \end{pmatrix} \quad \vec{y} = \alpha_1 \vec{x}_1 + \alpha_2 \vec{x}_2$$

$$\begin{cases} \langle x_1 | y \rangle = \alpha_1 \langle x_1 | x_1 \rangle + \alpha_2 \langle x_1 | x_2 \rangle \\ \langle x_2 | y \rangle = \alpha_1 \langle x_2 | x_1 \rangle + \alpha_2 \langle x_2 | x_2 \rangle \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} i = \alpha_1 \cdot 2 + \alpha_2 (1+i) \\ i = \alpha_1 (-i) + \alpha_2 \cdot 2 \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{1+i}{2} \\ \alpha_2 = \frac{-1+i}{2} \end{cases}$$

⑦ By direct inspection, the orthogonal pairs are: \vec{x}_1 and \vec{x}_3 ,
 \vec{x}_2 and \vec{x}_3 .

⑧ Consider an unknown vector $[a \ b \ c]^T = \vec{x}_3$

We must have:

$$\begin{cases} \langle \vec{x}_1 | \vec{x}_3 \rangle = 0 \\ \langle \vec{x}_2 | \vec{x}_3 \rangle = 0 \end{cases} \Rightarrow \begin{cases} a + b + c = 0 \\ a - b = 0 \end{cases}$$

Not enough equations, but that is because the vector can have any length. Let's pick $a = 1$. Then $b = 1$, $c = -2$.

$$\vec{x}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \quad |\vec{x}_3| = \sqrt{1+1+4} = \sqrt{6}.$$

And therefore the orthonormal basis is:

$$\left\{ \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}.$$