Week 18 workshop exercises

1. Find the norms and the scalar products of the following pairs of vectors:

   (a) $\vec{x} = \begin{pmatrix} 1 & 4 & 0 & 2 \end{pmatrix}^{\mathrm{T}}$ and $\vec{y} = \begin{pmatrix} 2 & -2 & 1 & 3 \end{pmatrix}^{\mathrm{T}}$;

   (b) $\vec{x} = \begin{pmatrix} 0 & i & 0 & -i \end{pmatrix}^{\mathrm{T}}$ and $\vec{y} = \begin{pmatrix} 0 & 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}$;

   (a) $\vec{x} = \begin{pmatrix} 1 & 0 & 1 & 0 \end{pmatrix}^{\mathrm{T}}$ and $\vec{y} = \begin{pmatrix} 0 & i & 0 & i \end{pmatrix}^{\mathrm{T}}$;

2. Can a system of three two-dimensional vectors be linearly independent? Construct a formal proof for your answer.

3. By checking the six properties of the vector space given in W18L1, demonstrate that the set of all square 3×3 matrices is… also a vector space. What is the dimension of this space? What other objects can you think of that would satisfy those six properties?

4. Explain why the following set of vectors is linearly independent:

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad \vec{x}_4 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}$$

5. The four vectors in the previous problem are an orthonormal basis set of the four-dimensional complex vector space. Find the expansion coefficients of $\begin{pmatrix} 3 & 7i & 8 & 0 \end{pmatrix}^{\mathrm{T}}$ in this basis set.

6. Given the following basis (not orthogonal, not normalised):

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

   find the expansion coefficients for $y = \begin{pmatrix} i & 0 \end{pmatrix}^{\mathrm{T}}$.

7. Which pairs are orthogonal among the following vectors?

$$\vec{x}_1 = \begin{pmatrix} 1 \\ 2 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{x}_2 = \begin{pmatrix} 4 \\ 0 \\ 4 \\ 0 \end{pmatrix}, \quad \vec{x}_3 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ -1 \end{pmatrix}, \quad \vec{x}_4 = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},$$

8. Find a vector in $\mathbb{R}^3$ that is orthogonal to $\begin{pmatrix} 1 & 1 & 1 \end{pmatrix}^{\mathrm{T}}$ and $\begin{pmatrix} 1 & -1 & 0 \end{pmatrix}^{\mathrm{T}}$. Produce an orthonormal basis from these three vectors.