Week 24 workshop exercises

1. Find the following norms and scalar products in the $x \in [-1,1]$ interval:

   (a) $\|x\|$ (b) $\|2^x\|$ (c) $\|\cos(\pi x/2)\|$

   (d) $\langle 1|x\rangle$ (e) $\langle x|2x^2-1\rangle$ (f) $\langle 2x^2-1|4x^3-3x\rangle$

2. Normalize the following functions on $x \in [-\pi,\pi]$ (find the value of N that would give a unit norm):

   (a) $f(x) = N$ (b) $f(x) = N\sin x$ (c) $f(x) = N^{-1}\cos(2x)$

   (d) $f(x) = N\sin^2 x$ (e) $f(x) = N(i - x/\pi)$ (f) $f(x) = Ne^{-4ix}$

3. The square of the norm of the wavefunction in quantum mechanics is equal to the probability of finding the system anywhere within its allowed range of states. Calculate the norms of the following wavefunctions in the indicated intervals and comment on the values that you obtain

   (a) $\psi(x,y) = \dfrac{1}{\sqrt{\pi}}\exp\left[-\dfrac{x^2+y^2}{2}\right] \qquad x,y \in [-\infty,\infty]$

   (b) $\psi(x) = \sqrt{\dfrac{2}{\pi}}\dfrac{1}{x^2+1} \qquad x \in [-\infty,\infty]$

   Hints: in (a), consider using polar coordinates for integration; in (b), use $x = \tan\varphi$ as a substitution.

4. Determine whether the following sets of functions are linearly independent (*i.e.* no function may be expressed as a linear combination of the two others):

   (a) $\{x, 1+x, 1-x\}$ (b) $\{\cos x, \cos 2x, \cos 3x\}$

   (c) $\{0, x\ln x, \arctan x, e^x\}$ (d) $\{\ln(x), \ln(x^2), \ln(x^3)\}$

5. Demonstrate that the following set of complex exponentials (called *plane waves*)

   $$g_k(x) = e^{ikx}/\sqrt{2\pi} \qquad k = 0, \pm 1, \pm 2, \ldots$$

   is orthonormal on the $x \in [-\pi,\pi]$ interval.

   (a) Find the expansion coefficients for the following functions in the basis set $\{g_{-1}, g_0, g_{+1}\}$ for and demonstrate that their scalar products with all other plane waves are zero:

   (i) $f(x) = \cos x$ (ii) $f(x) = \sin x$ (iii) $f(x) = 1$

   (b) Find the matrix representation for the second derivative operator on the $[-\pi,\pi]$ interval in the plane wave basis set. Specify all ∞^2 elements.