Week 26 problem set

1. Demonstrate that $Y_{2,0}(\theta, \varphi)$ given in Table 1 of Lecture 1:
   
   (a) satisfies Equation 18 from the same lecture,
   
   (b) satisfies $\hat{L}_z Y_{2m}(\theta, \varphi) = m Y_{2m}(\theta, \varphi)$ where $\hat{L}_z = -i \partial / \partial \varphi$.

2. Demonstrate that $Y_{2,1}(\theta, \varphi)$ and $Y_{1,0}(\theta, \varphi)$ given in Table 1 of Lecture 1 are mutually orthogonal and normalised with respect to the following norm and scalar product:

   \[
   \langle Y_{ab}(\theta, \varphi) \vert Y_{cd}(\theta, \varphi) \rangle = \frac{2\pi}{\int_0^\infty \int_0^{2\pi} Y_{ab}^*(\theta, \varphi) Y_{cd}(\theta, \varphi) \sin \theta \, d\theta \, d\varphi}
   \]

   \[
   \| Y_{ab}(\theta, \varphi) \| = \sqrt{\langle Y_{ab}(\theta, \varphi) \vert Y_{ab}(\theta, \varphi) \rangle}
   \]

3. Try to prove that the eigenvectors of Hermitian matrices ($A^\dagger = A$ where dagger stands for conjugate-transpose) corresponding to different eigenvalues are orthogonal.

4. The probability of finding the electron somewhere inside a shell of radius $r$ and of thickness $dr$ is proportional to $r^2 |R_{nl}(r)|^2 \, dr$. The factor $|R_{nl}(r)|^2$ gives the probability density at radius $r$, and the factor $r^2 \, dr$ is proportional to the volume of the shell. Prove the statement made in Equation 8 of Lecture 2 by following through the optimisation procedure.

5. Calculate the expectation value $\langle r \rangle = \langle \psi \vert r \vert \psi \rangle$ for the ground state of hydrogen. Do not forget the angular integral and the spherical coordinate Jacobian.

6. Calculate the wavelength of the transition between the $n = 1$ and $n = 2$ states of hydrogen (this is called the Lyman-\(\alpha\) line).

7. Calculate the wavelength of the transition between the $n = 2$ and $n = 3$ states of hydrogen (this is called the Balmer-\(\alpha\) line).

8. The energy of a hydrogen-like atom (one electron, but a different nuclear charge) is given by

   \[
   E_n = -\frac{\mu e^4 Z^2}{2(4\pi\varepsilon_0)^2 \hbar^2 n^2}
   \]

   where $Z$ is an integer giving the nuclear charge in units of proton charge. Calculate the wavelength of the Balmer-\(\alpha\) line in the He$^+$ ion. Remember to update the reduced mass.

9. The energy levels of a one-electron atom are proportional to the reduced mass of the system, and so there will be a small energy difference between the levels of hydrogen and deuterium. As a result, the spectral lines of deuterium will be slightly shifted in frequency relative to those of hydrogen. This is known as isotope shift. Calculate the isotope shift in frequency units for the Lyman-\(\alpha\) line of hydrogen and deuterium.

10. Positronium is a bound system of an electron and a positron. What is the wavelength of the Balmer-\(\alpha\) transition in positronium?

11. Muonic hydrogen is a bound system of a muon orbiting a proton. Muons are heavier than electrons. How would this influence the size of the resulting “atom” in its ground state?