CHEM1047 – Week 1 Lecture 2 – Complex numbers

☐ *Chapter 8 of Steiner, "The Chemistry Maths Book", 2nd edition.*

☐ *Volume 2, Chapter 2 of Cockett and Doggett, "Maths for Chemists".*

1. Formal definition of a number

There are many types of numbers in mathematics: integers, rational numbers, real numbers, *etc.* They share some features and provide a good example of formal definitions and notation used in physical sciences. A *field* is formally defined as a set $\mathbb{F}$ with two operations, called addition ("+") and multiplication ("·"), that satisfy the following properties:

1. $\mathbb{F}$ is closed under addition and multiplication:

$$\forall a,b\in\mathbb{F},\quad a+b\in\mathbb{F}\quad\text{and}\quad a\cdot b\in\mathbb{F}$$

   This should be read in the following way: "for any elements a and b belonging to $\mathbb{F}$, their sum and their product also belong to $\mathbb{F}$".

2. Addition and multiplication operations are *associative*:

$$\forall a,b,c\in\mathbb{F}\quad (a+b)+c=a+(b+c)\quad\text{and}\quad (a\cdot b)\cdot c=a\cdot(b\cdot c)$$

3. Addition and multiplication operations are *commutative*:

$$\forall a,b\in\mathbb{F}\quad a+b=b+a\quad\text{and}\quad a\cdot b=b\cdot a$$

4. $\mathbb{F}$ contains a unique zero element and a unique unit element:

$$\begin{gathered}\exists 0\in\mathbb{F},\quad \forall a\in\mathbb{F}\quad a+0=a\\ \exists 1\in\mathbb{F},\quad \forall a\in\mathbb{F}\quad a\cdot 1=a\end{gathered}$$

   The first line should be read in the following way: "there exists an element 0 in $\mathbb{F}$, such that, for any element a belonging to $\mathbb{F}$, $a+0=a$". The second line is: "there exists an element 1 in $\mathbb{F}$, such that, for any element a belonging to $\mathbb{F}$, $a\cdot 1=a$".

5. Each element in $\mathbb{F}$ has a unique additive inverse in $\mathbb{F}$:

$$\forall a\in\mathbb{F}\quad \exists(-a)\in\mathbb{F},\quad a+(-a)=0$$

   and a unique multiplicative inverse in $\mathbb{F}$:

$$\forall a\in\mathbb{F}\quad \exists a^{-1}\in\mathbb{F},\quad a\cdot a^{-1}=1$$

6. Multiplication is *distributive* over addition (*i.e.* the brackets are opened in the usual way):

$$\forall a,b,c\in\mathbb{F}\quad a\cdot(b+c)=a\cdot b+a\cdot c$$

Not all sets of numbers are fields (*e.g.* integers $\mathbb{Z}$ are not), but three important fields are $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers) and $\mathbb{C}$ (complex numbers).

Example 1: prove that the set of all integers is not a field.

Solution: multiplicative inverse of an integer is not necessarily an integer – for example, 1/3. However, property 5 above requires the inverse to be of the same type. Integers do not therefore satisfy the definition of a field.

2. Complex numbers

A *complex number* is a number that can be expressed as $a+bi$, where $a,b\in\mathbb{R}$ and $i^2=-1$. The two parts a and b are called *real* and *imaginary* parts, and denoted

$$a=\mathrm{Re}(a+bi) \qquad b=\mathrm{Im}(a+bi) \tag{1}$$

Complex numbers may be added in the usual way:

$$(a_1+b_1i)+(a_2+b_2i)=a_1+b_1i+a_2+b_2i=(a_1+a_2)+(b_1+b_2)i \tag{2}$$

and multiplied in the usual way:

$$(a_1+b_1i)(a_2+b_2i)=a_1a_2+a_1b_2i+a_2b_1i+b_1b_2i^2=(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i \tag{3}$$

An operation specific to complex numbers is *conjugation* (all instances of i flip the sign):

$$(a+bi)^*=a-bi \tag{4}$$

This operation is useful for dividing complex numbers:

$$\begin{aligned}\frac{a_1+b_1i}{a_2+b_2i}&=\frac{(a_1+b_1i)(a_2-b_2i)}{(a_2+b_2i)(a_2-b_2i)}=\frac{(a_1a_2+b_1b_2)+(a_2b_1-a_1b_2)i}{a_2^2+b_2^2}=\\&=\left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right)+\left(\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right)i\end{aligned} \tag{5}$$

Here both sides of the fraction were multiplied by the conjugate of the denominator. This makes the denominator real and allows the expression to be simplified.

Complex numbers appeared in the 16th century (in the works of Tartaglia, Cardano, and Bombelli) as formal solutions to polynomial equations, and later took on a life of their own when it transpired that many fundamental functions in physics are complex.

Example 2: simplify the following expressions

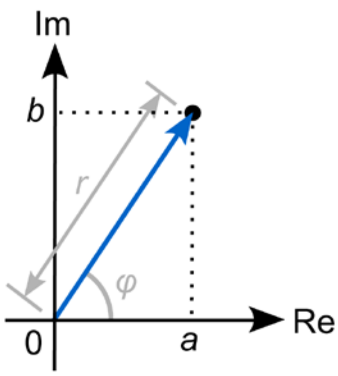
$$(2+5i)(3-2i) \qquad (2+5i)+(3-2i) \qquad (2+5i)/(3-2i)$$

Solution: following the recipes given above eventually yields $16+11i$, $5+3i$, and $-(4/13)+(19/13)i$.

3. Cartesian and polar representations of complex numbers

Because a complex number incorporates two independently varying real numbers, it can be interperted as another common two-element object – a *vector* in two dimensions:

$$z=a+bi \qquad \Leftrightarrow \qquad \vec{z}=\begin{bmatrix}a & b\end{bmatrix}$$



The length of this vector $|\vec{z}|=\sqrt{a^2+b^2}$ is called the *absolute value* (or *amplitude*) of the complex number $z=a+bi$. Amplitude can also be expressed in terms of the number and its conjugate: $|z|=\sqrt{z^*z}$.

If the same vector is expressed in polar coordinates, the absolute value is the length of the vector. The polar angle φ is called *phase angle*. For vectors:

$$\vec{r}=\begin{bmatrix}x & y\end{bmatrix} \qquad |\vec{r}|=\sqrt{x^2+y^2} \qquad \varphi=\mathrm{atan}(y/x) \qquad \vec{r}=|\vec{r}|\begin{bmatrix}\cos\varphi & \sin\varphi\end{bmatrix} \tag{6}$$

For complex numbers:

$$z = a + bi \qquad |z| = \sqrt{a^2 + b^2} \qquad \varphi = \operatorname{atan}(b/a) \qquad z = |z|(\cos\varphi + i\sin\varphi) \tag{7}$$

The close analogy with two-dimensional vectors is apparent. The figure above is called *Argand diagram*.

4. Complex exponentials

The combination $\cos\varphi + i\sin\varphi$ has a special status in mathematics. Consider (the proof will be given a few weeks' time) power series expansions for sine and cosine functions:

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + \ldots \qquad \sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} - \frac{\varphi^7}{7!} + \ldots \tag{8}$$

The cosine part contains all the even powers of φ, and the sine part all the odd powers. If we put them together as $\cos\varphi + i\sin\varphi$ and simplify the expression, we would get:

$$\cos\varphi + i\sin\varphi = \frac{(i\varphi)^0}{0!} + \frac{(i\varphi)^1}{1!} + \frac{(i\varphi)^2}{2!} + \frac{(i\varphi)^3}{3!} + \ldots$$

A quick look into a power series reference book would convince us that we are looking at a power series expansion of the exponential function:

$$e^x = \frac{x^0}{0!} + \frac{x^1}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \tag{9}$$

Therefore (*Euler's formula)*:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi \tag{10}$$

This relation is very powerful. The reciprical relations

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i} \tag{11}$$

are useful in trigonometric simplifications because they convert trigonometric functions into exponentials that are easier to simplify. The special case of $\varphi = \pi$

$$\cos\pi + i\sin\pi = e^{i\pi} \quad \Rightarrow \quad e^{i\pi} + 1 = 0$$

illustrates the deep connection between e, π, i, 1, and 0.

Example 3: find polar representations of the following complex numbers

$$2 \qquad 4i \qquad 1+i \qquad 1-\sqrt{3}i$$

Solution: direct application of Equations (7) yields

$$\begin{cases} r = 2 \\ \varphi = 0 \end{cases} \qquad \begin{cases} r = 4 \\ \varphi = \pi/2 \end{cases} \qquad \begin{cases} r = \sqrt{2} \\ \varphi = \pi/4 \end{cases} \qquad \begin{cases} r = 2 \\ \varphi = -\pi/3 \end{cases}$$

Example 4: separate the frequency response function of an electrical oscillator

$$f(\omega) = 1/(1 + i\omega)$$

into its real and imaginary parts by multiplying the numerator and the denominator by the complex conjugate of the denominator. Assume that the argument ω is real.

Solution: this is a case of the division operation. Applying the recipe given in Equation (5) yields

$$f(\omega)=\frac{1}{1+i\omega}=\frac{1-i\omega}{(1+i\omega)(1-i\omega)}=\frac{1-i\omega}{1+\omega^2}=\left(\frac{1}{1+\omega^2}\right)-\left(\frac{\omega}{1+\omega^2}\right)i$$

where the real and the imaginary parts can now be identified as

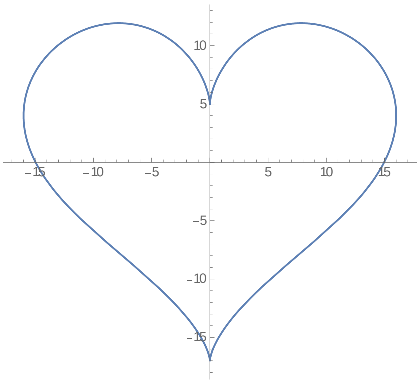
$$\mathrm{Re}\left[f(\omega)\right]=\frac{1}{1+\omega^2} \qquad \mathrm{Im}\left[f(\omega)\right]=-\frac{\omega}{1+\omega^2}$$

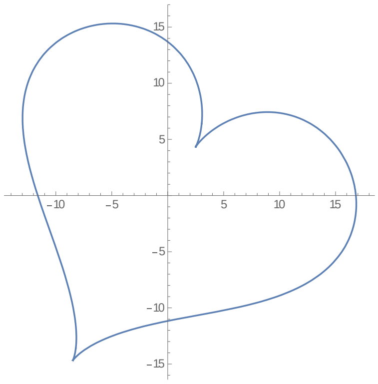
4. Complex numbers as scaling and rotation operators

Consider a complex-valued function $f(t)$ of time t. As an example, let us take the following:

$$\begin{aligned}\mathrm{Re}\left[f(t)\right]&=16\sin^3(t)\\ \mathrm{Im}\left[f(t)\right]&=13\cos(t)-5\cos(2t)-2\cos(3t)-\cos(4t)\end{aligned}$$

In the Cartesian representation, its plot is a curve in two dimensions, parameterised by t.





If we multiply the function by $Ae^{i\varphi}$, where A and φ are real numbers, the following would happen:

$$A(\cos\varphi+i\sin\varphi)\Big(\mathrm{Re}\left[f(t)\right]+i\,\mathrm{Im}\left[f(t)\right]\Big)=$$
$$=A\Big\{\Big(\mathrm{Re}\left[f(t)\right]\cos\varphi-\mathrm{Im}\left[f(t)\right]\sin\varphi\Big)+\Big(\mathrm{Im}\left[f(t)\right]\cos\varphi+\mathrm{Re}\left[f(t)\right]\sin\varphi\Big)i\Big\}$$

Two things occurred – the function got scaled by A, and its real and imaginary parts got mixed:

$$\begin{aligned}\mathrm{Re}\left[e^{i\varphi}f(t)\right]&=\mathrm{Re}\left[f(t)\right]\cos\varphi-\mathrm{Im}\left[f(t)\right]\sin\varphi\\ \mathrm{Im}\left[e^{i\varphi}f(t)\right]&=\mathrm{Re}\left[f(t)\right]\sin\varphi+\mathrm{Im}\left[f(t)\right]\cos\varphi\end{aligned}$$

If we identify $\mathrm{Re}\left[f(t)\right]$ with x and $\mathrm{Im}\left[f(t)\right]$ with y, we can recognise the rotation operation:

$$\begin{aligned}x'&=x\cos\varphi-y\sin\varphi\\ y'&=x\sin\varphi+y\cos\varphi\end{aligned}$$

Therefore the multiplication of some object by a complex number $Ae^{i\varphi}$ scales it by A and rotates it clockwise by φ. This operation finds a lot of use in computer graphics – a multiplication by a complex number is faster than evaluation of trigonometric functions. When billions of rotation operations must be performed per second, this is a significant advantage.