## CHEM1030 - Week 2 Lecture - complex numbers

*Chapter 8 of Steiner, "The Chemistry Maths Book", 2nd edition.*

### The field of complex numbers

The word *field* in mathematics refers to a set of elements that is closed with respect to two operations, called *addition* and *multiplication*, in which those operations individually are *associative* (the order of brackets does not matter), *commutative* (the order of operands does not matter), and *distributive* (brackets in mixed operations are opened in the usual way). There must additionally exist unique zero and unit elements, and for each element there must be a unique negative element and a unique inverse element. Not all sets of numbers are fields (*e.g.* integers are not), but three important fields that we need to know about are $\mathbb{Q}$ (rational numbers), $\mathbb{R}$ (real numbers) and $\mathbb{C}$ (complex numbers).

A complex number is a number that can be expressed as $a+bi$, where $a,b\in\mathbb{R}$ and $i^2=-1$. The two parts $a$ and $b$ are called *real* and *imaginary* part respectively and denoted

$$a=\mathrm{Re}(a+bi) \qquad b=\mathrm{Im}(a+bi) \tag{1}$$

Complex numbers may be added and multiplied in the usual way:

$$\begin{aligned}(a_1+b_1i)+(a_2+b_2i)&=a_1+b_1i+a_2+b_2i=(a_1+a_2)+(b_1+b_2)i\\(a_1+b_1i)(a_2+b_2i)&=a_1a_2+a_1b_2i+a_2b_1i+b_1b_2i^2=(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i\end{aligned} \tag{2}$$

Division operation requires more care:

$$\frac{a_1+b_1i}{a_2+b_2i}=\frac{(a_1+b_1i)(a_2-b_2i)}{(a_2+b_2i)(a_2-b_2i)}=\frac{(a_1a_2+b_1b_2)+(a_2b_1-a_1b_2)i}{a_2^2+b_2^2}=\left(\frac{a_1a_2+b_1b_2}{a_2^2+b_2^2}\right)+\left(\frac{a_2b_1-a_1b_2}{a_2^2+b_2^2}\right)i \tag{3}$$

An operation specific to complex numbers is *conjugation* (all instances of $i$ flip the sign):

$$(a+bi)^*=a-bi \tag{4}$$

Complex numbers initially arose in 16th century as formal solutions to some polynomial equations, but later took on a life of their own when it transpired that many fundamental functions in physics are complex-valued. The counterintuitive nature of their practical interpretation, amongst other similar things, has kept "philosophers of science" in business ever since.

<u>Example 1</u>: simplify the following expressions

$$(2+5i)(3-2i) \qquad (2+5i)+(3-2i) \qquad (2+5i)/(3-2i)$$

<u>Answers</u>:

$$16+11i \qquad 5+3i \qquad -(4/13)+(19/13)i$$

### Cartesian and polar representations of complex numbers

Because a complex number incorporates two independently varying real numbers, it can be associated with another common two-element object – a vector in two dimensions:

$$c=a+bi \qquad \Leftrightarrow \qquad \vec{c}=\begin{pmatrix}a & b\end{pmatrix}$$

The length of this vector $r=\sqrt{a^2+b^2}$ is called the *modulus* or *amplitude* of the complex number $c=a+bi$. It can also be expressed in terms of the complex number and its conjugate: $|c|=\sqrt{c^*c}$. If the same vector is expressed in polar coordinates, the modulus corresponds to the length of the vector and the corresponding angle $\varphi$ is called the *phase* of the complex number:

$$\vec{r} = \begin{pmatrix} x & y \end{pmatrix} \qquad r = \sqrt{x^2 + y^2} = |\vec{r}| \qquad \varphi = \text{atan}(y/x) \qquad \vec{r} = \begin{pmatrix} r\cos\varphi & r\sin\varphi \end{pmatrix}$$

$$c = a + bi \qquad A = \sqrt{a^2 + b^2} = |c| \qquad \varphi = \text{atan}(b/a) \qquad c = A(\cos\varphi + i\sin\varphi)$$

The close analogy with two-dimensional vectors is apparent. The combination $\cos\varphi + i\sin\varphi$ has a special status in mathematics. Consider the corresponding Taylor series:

$$\cos\varphi = 1 - \frac{\varphi^2}{2!} + \frac{\varphi^4}{4!} - \frac{\varphi^6}{6!} + O\left[\varphi^8\right] = \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(i\varphi)^{2n}}{(2n)!}$$

$$\sin\varphi = \varphi - \frac{\varphi^3}{3!} + \frac{\varphi^5}{5!} + O\left[\varphi^7\right] = \sum_{n=0}^{\infty} \frac{(-1)^n \varphi^{2n+1}}{(2n+1)!} = \frac{1}{i}\sum_{n=0}^{\infty} \frac{(i\varphi)^{2n+1}}{(2n+1)!}$$

The cosine series contains all the even powers of $i\varphi$ divided by the corresponding factorials and the sine series contains all the odd powers with their corresponsing factorials. If we put the two series together (taking care to get rid of the $1/i$ term in front of the sine series), we get:

$$\cos\varphi + i\sin\varphi = \sum_{n=0}^{\infty} \frac{(i\varphi)^{2n}}{(2n)!} + \sum_{n=0}^{\infty} \frac{(i\varphi)^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(i\varphi)^n}{n!}$$

in which the Taylor series for the exponential function may be recognized. Therefore:

$$e^{i\varphi} = \cos\varphi + i\sin\varphi$$

This relation (known as *Euler's formula*) is very powerful. In particular, the reciprical relations

$$\cos\varphi = \frac{e^{i\varphi} + e^{-i\varphi}}{2} \qquad \sin\varphi = \frac{e^{i\varphi} - e^{-i\varphi}}{2i}$$

are useful in trigonometric simplifications because they convert trigonometric expressions into exponential expressions that are easier to simplify. The special case of $\varphi = \pi$

$$\cos\pi + i\sin\pi = e^{i\pi} \quad \Rightarrow \quad e^{i\pi} + 1 = 0$$

illustrates the deep connection between $e$, $\pi$, $i$, $1$ and $0$.

<u>Example 2</u>: compute polar representations of the following complex numbers

$$2 \qquad 4i \qquad 1+i \qquad 1-\sqrt{3}i$$

<u>Answers</u>:

$$\begin{cases} r = 2 \\ \varphi = 0 \end{cases} \qquad \begin{cases} r = 4 \\ \varphi = \pi/2 \end{cases} \qquad \begin{cases} r = \sqrt{2} \\ \varphi = \pi/4 \end{cases} \qquad \begin{cases} r = 2 \\ \varphi = -\pi/3 \end{cases}$$

<u>Example 3</u>: separate the frequency response function of an electrical oscillator

$$f(\omega) = \frac{1}{1 + i\omega}$$

into its real and imaginary parts by multiplying the numerator and the denominator by the complex conjugate of the denominator. Assume that the argument $\omega$ is real.

<u>Solution</u>:

$$f(\omega)=\frac{1}{1+i\omega}=\frac{1-i\omega}{(1+i\omega)(1-i\omega)}=\frac{1-i\omega}{1+\omega^2}=\left(\frac{1}{1+\omega^2}\right)-\left(\frac{\omega}{1+\omega^2}\right)i$$

$$\mathrm{Re}\left[f(\omega)\right]=\frac{1}{1+\omega^2} \qquad \mathrm{Im}\left[f(\omega)\right]=-\frac{\omega}{1+\omega^2}$$

## Complex numbers as 2D scaling and rotation operators

Consider some complex-valued function $f(t)$ of time or any other parameter that we would call $t$. As an example, let us take the following:

$$\mathrm{Re}\left[f(t)\right]=16\sin^3(t)$$
$$\mathrm{Im}\left[f(t)\right]=13\cos(t)-5\cos(2t)-2\cos(3t)-\cos(4t)$$

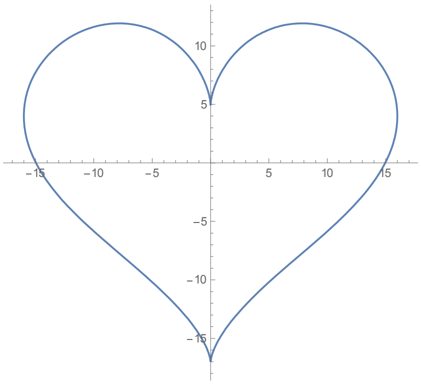
In the Cartesian representation its plot would look as shown below – a curve in two dimensions, parameterized by $t$. If we multiply the function by $Ae^{i\varphi}$, where $A$ and $\varphi$ are real numbers, the following would happen:

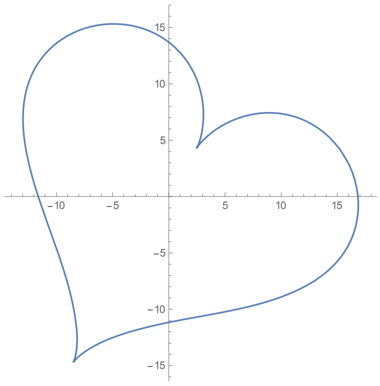
$$A(\cos\varphi+i\sin\varphi)\left(\mathrm{Re}\left[f(t)\right]+i\,\mathrm{Im}\left[f(t)\right]\right)=$$
$$=A\left\{\left(\mathrm{Re}\left[f(t)\right]\cos\varphi-\mathrm{Im}\left[f(t)\right]\sin\varphi\right)+\left(\mathrm{Im}\left[f(t)\right]\cos\varphi+\mathrm{Re}\left[f(t)\right]\sin\varphi\right)i\right\}$$

Two things have happened – the function got scaled by $A$ and its real and imaginary parts got mixed. The new real and imaginary parts are:

$$\mathrm{Re}\left[e^{i\varphi}f(t)\right]=\mathrm{Re}\left[f(t)\right]\cos\varphi-\mathrm{Im}\left[f(t)\right]\sin\varphi$$
$$\mathrm{Im}\left[e^{i\varphi}f(t)\right]=\mathrm{Re}\left[f(t)\right]\sin\varphi+\mathrm{Im}\left[f(t)\right]\cos\varphi$$

If we identify $\mathrm{Re}\left[f(t)\right]$ with $x$ and $\mathrm{Im}\left[f(t)\right]$ with $y$, it is easy to recognize the rotation operation. It does therefore appear that the multiplication of a complex-valued function $f(t)$ by a complex number $Ae^{i\varphi}$ scales the plot of its Cartesian representation by $A$ and rotates it counterclockwise by $\varphi$.




## *Week 22 workshop exercises*

Steiner, 2nd edition: section 8.8, problems 1, 4, 6-8, 12, 14, 16-19, 28, 30, 32.

## *Extra difficulty exercises for the brave*

Steiner, 2nd edition: section 8.8, problems 48-50.