CHEM1033 - Week 4 Lecture - Derivatives and differentiation II

Section 4.6 of Steiner, "The Chemistry Maths Book", 2nd edition.
Chapter 18 of Monk and Munro, "Maths for Chemistry", 2nd edition.

1. Product differentiation rule

The limit definition discussed in the previous lecture is in practice hard to use. A more convenient way is to compile a table of derivatives for common functions and their combinations. We have already demonstrated that the differentiation operation is linear:

$$\left[\alpha f(x)+\beta g(x)\right]'=\alpha f'(x)+\beta g'(x) \tag{1}$$

for any functions $f(x)$, $g(x)$ and any constants α, β. This relation allows us to differentiate sums and differences of functions. To derive the corresponding rule for products of functions, we need to make a few preliminary observations. The limit definition of the derivative:

$$f'(x)=\lim_{\Delta x\to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x} \tag{2}$$

holds strictly for infinitely small values of Δx. For finite values of Δx we would have:

$$f'(x)=\frac{f(x+\Delta x)-f(x)}{\Delta x}+\varphi(x,\Delta x) \tag{3}$$

where the unknown function $\varphi(x,\Delta x)$ goes to zero when Δx goes to zero. After rearranging this formula to make $f(x+\Delta x)$ the subject, we obtain:

$$f(x+\Delta x)=f(x)+f'(x)\Delta x-\varphi(x,\Delta x)\Delta x \tag{4}$$

If we take some other function $g(x)$ and subject it to a similar treatment, we would obtain

$$g(x+\Delta x)=g(x)+g'(x)\Delta x-\psi(x,\Delta x)\Delta x \tag{5}$$

with some other unknown function $\psi(x,\Delta x)$ which also goes to zero when Δx goes to zero. That's it for the preliminary observations. Now for the derivative of the product of $f(x)$ and $g(x)$:

$$\begin{aligned}\left[f(x)g(x)\right]' &= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x)-f(x)g(x)}{\Delta x}=\\ &=\lim_{\Delta x\to 0}\frac{\left[f(x)+f'(x)\Delta x-\varphi(x,\Delta x)\Delta x\right]\left[g(x)+g'(x)\Delta x-\psi(x,\Delta x)\Delta x\right]-f(x)g(x)}{\Delta x}=\\ &=\lim_{\Delta x\to 0}\frac{f'(x)g(x)\Delta x+f(x)g'(x)\Delta x+[...]}{\Delta x}=f'(x)g(x)+f(x)g'(x)\end{aligned} \tag{6}$$

where all terms in square brackets $[...]$ have a zero limit when Δx goes to zero (inspect them as an exercise). Therefore, the rule for the differentiation of products of functions is:

$$\left[f(x)g(x)\right]'=f'(x)g(x)+f(x)g'(x) \tag{7}$$

2. Fraction differentiation rule

The problem of differentiating the ratio of two functions may be partially reduced to the equation we have just obtained for the derivative of a product:

$$\left[\frac{f(x)}{g(x)}\right]' = \left[f(x)\frac{1}{g(x)}\right]' = f'(x)\frac{1}{g(x)} + f(x)\left[\frac{1}{g(x)}\right]' \tag{8}$$

and so the question is reduced to finding out what the derivative of $1/g(x)$ is. That is quite easy:

$$\begin{aligned}\left[\frac{1}{g(x)}\right]' &= \lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}\left(\frac{1}{g(x+\Delta x)}-\frac{1}{g(x)}\right)\right] = \lim_{\Delta x\to 0}\left[\frac{1}{\Delta x}\left(\frac{g(x)-g(x+\Delta x)}{g(x+\Delta x)g(x)}\right)\right] = \\ &= -\lim_{\Delta x\to 0}\left[\frac{g(x+\Delta x)-g(x)}{\Delta x}\frac{1}{g(x+\Delta x)g(x)}\right] = -\frac{g'(x)}{g(x)^2}\end{aligned} \tag{9}$$

We can now continue the simplification of Equation (8):

$$\left[\frac{f(x)}{g(x)}\right]' = f'(x)\frac{1}{g(x)} - f(x)\frac{g'(x)}{g(x)^2} = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2} \tag{10}$$

Therefore, the rule for the differentiation of fractions is:

$$\left[\frac{f(x)}{g(x)}\right]' = \frac{f'(x)g(x)-f(x)g'(x)}{g(x)^2} \tag{11}$$

Details of the derivations presented in this section are not examinable, but you are expected to memorize Equations (1), (7) and (11).

3. Derivatives of common functions

This section contains a fairly boring enumeration of common function derivatives – we only need to do it once. The simplest case is the derivative of a constant:

$$[\alpha]' = \lim_{\Delta x\to 0}\frac{\alpha-\alpha}{\Delta x} = 0 \tag{12}$$

The next most popular function is the linear function:

$$f(x) = ax + b \tag{13}$$

We can already handle sums and constants, and so the question is about the derivative of x:

$$[x]' = \lim_{\Delta x\to 0}\frac{x+\Delta x - x}{\Delta x} = \lim_{\Delta x\to 0}\frac{\Delta x}{\Delta x} = 1 \tag{14}$$

The next step in the complexity ladder is the polynomial, or rather its individual terms that contain powers of x. We can try to use the product rule here:

$$\left[x^n\right]' = \left[xx^{n-1}\right]' = [x]'\,x^{n-1} + x\left[x^{n-1}\right]' = x^{n-1} + x\left[x^{n-1}\right]' \tag{15}$$

This is a recursively defined sequence. We know from Equation (14) that $[x]' = 1$, therefore:

$$\begin{aligned}&[x]' = 1 \\ &\left[x^2\right]' = x + x[x]' = 2x \\ &\left[x^3\right]' = x^2 + x\left[x^2\right]' = x^2 + x[2x] = 3x^2 \\ &\left[x^4\right]' = x^3 + x\left[x^3\right]' = x^2 + x\left[3x^2\right] = 4x^3\end{aligned} \tag{16}$$

and so on. It is easy to see by direct inspection that $\left[x^n\right]' = nx^{n-1}$. We would not prove this statement here, but this formula is also valid for negative and fractional powers.

Similar procedures may be applied to other elementary functions, with the following results:

Function	**Derivative**
c (constant)	0
x^n	nx^{n-1}
e^x	e^x
a^x	$a^x \ln a$
$\sin x$	$\cos x$
$\cos x$	$-\sin x$
$\ln x$	$1/x$
$\log_b x$	$1/(x \ln b)$

You are expected to memorize this table. Together with Equations (1), (7) and (11), this table allows us to differentiate any additive or multiplicative combination of functions, for example:

$$\left[\frac{\cos(x)\ln(x)}{x^2}\right]' = \frac{\left[\cos(x)\ln(x)\right]' x^2 - \cos(x)\ln(x)\left[x^2\right]'}{x^4} =$$

$$= \frac{\left(\left[\cos(x)\right]'\ln(x) + \cos(x)\left[\ln(x)\right]'\right)x^2 - 2x\cos(x)\ln(x)}{x^4} =$$

$$= \frac{\left(-\sin(x)\ln(x) + \dfrac{\cos(x)}{x}\right)x^2 - 2x\cos(x)\ln(x)}{x^4} = \frac{\left(1-2\ln(x)\right)\cos(x)}{x^3} - \frac{\sin(x)\ln(x)}{x^2}$$

A lot of practice is required before such calculations become easy and natural. Despite conspicuous lack of any chemistry in the two sections above, all of this will be used heavily in your second year and should therefore be studied with diligence.

3. Alternative notations for the derivative

The limit definition of the derivative involves the ratio of the increment in the function divided by the increment of the argument that caused it. This fact has led to an alternative notation for the derivative that will be useful later because it can actually be interpreted as a fraction:

$$g'(x) \equiv \frac{dg(x)}{dx} \equiv \frac{d}{dx} g(x) \tag{17}$$

The d/dx prefix is called differentiation operator, it is an instruction to calculate the derivative of whatever occurs in front of it in the same sense as the prime symbol is an instruction to compute the derivative of whatever comes before it. In this notation, the problem we have just solved above would be

$$\frac{d}{dx}\left[\frac{\cos(x)\ln(x)}{x^2}\right]$$

We will see more of this notation when we consider integrals and differential equations. A less common notation used in physics for derivatives with respect to time is $\dot{g}(t) \equiv g'(t) \equiv dg(t)/dt$.

Week 5 workshop exercises

Monk and Munro, 2nd edition: self-tests 18.1-18.4; problems 18.4, 18.6, 18.8, 18.10.
Steiner, 2nd edition: Section 4.13 (23-27, 29-55).

Extra difficulty exercises for the brave

Steiner, 2nd edition: Section 4.13 (69-71).