

CHEG2002 - Week 21, Lecture 1 - Laplace transform

Reading: Seborg, Edgar, Mellichamp, and Doyle, "Process Dynamics and Control", 3rd ed., Chapter 3.

The Laplace transform, named after the French mathematician Pierre-Simon Laplace (1749-1827), has roots stretching back to the 18th century. Although Laplace did not directly invent the transform that bears his name, his work in probability theory and celestial mechanics has laid the groundwork.

The concept of transforming a function into a different domain for simplification and analysis was further developed in the early 20th century, notably through the work of Oliver Heaviside who applied it to solve differential equations in electrical engineering. Heaviside's methods initially lacked a rigorous mathematical foundation (which came later), but were instrumental in engineering applications.

1. Definition and basic properties

The Laplace transform $F(s)$ of a function $f(t)$, such that $f(t < 0) = 0$, is defined as:

$$F(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} f(t) e^{-st} dt \quad (1)$$

The following properties follow directly from the definition:

- (a) **Linearity.** For any two functions $f(t)$ and $g(t)$ that have a Laplace transform, and any two numbers α and β :

$$\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\} \quad (2)$$

- (b) **Time shift.** If $f(t)$ is shifted in time by t_0 , its Laplace transform is multiplied by $e^{-t_0 s}$:

$$\mathcal{L}\{f(t - t_0)\} = e^{-t_0 s} F(s) \quad (3)$$

- (c) **Time scaling.** If the time t is scaled by a factor α under the function, its Laplace transform is scaled by $1/\alpha$:

$$\mathcal{L}\{f(\alpha t)\} = \frac{1}{\alpha} F\left(\frac{s}{\alpha}\right) \quad (4)$$

- (d) **Differentiation.** The Laplace transform of the derivative $f'(t)$ with an initial value $f(0)$ has a simple expression via the Laplace transform of $f(t)$:

$$\mathcal{L}\{f'(t)\} = sF(s) - f(0) \quad (5)$$

- (e) **Integration.** The Laplace Transform of the integral of $f(t)$ from 0 to t is given by

$$\mathcal{L}\left\{\int_0^t f(\tau) d\tau\right\} = \frac{1}{s} F(s) \quad (6)$$

Similar properties (homework exercise) may be proven for second and higher derivatives, and for multiple integrals taken as functions of their last upper limit.

2. Examples of Laplace transforms

The following Laplace transforms may be performed by taking the integral in the definition:

- (a) **Constant function:** $\mathcal{L}\{C\} = C/s$.
- (b) **Power function:** $\mathcal{L}\{t^n\} = n!/s^{n+1}$ (proceed using integration by parts).
- (c) **Exponential function:** $\mathcal{L}\{e^{at}\} = 1/(s-a)$.
- (d) **Sine and cosine:** $\mathcal{L}\{\sin(\omega t)\} = \omega/(s^2 + \omega^2)$, $\mathcal{L}\{\cos(\omega t)\} = s/(s^2 + \omega^2)$ (proceed using Euler's expressions for sine and cosine via complex exponentials).

Many other standard formulas of this type may be found in textbooks.

3. Solving ordinary differential equations with Laplace transform

The differentiation property of the Laplace transform simplifies the process of solving linear ODEs by transforming the differential equations into algebraic equations. The general steps are:

1. Applying the Laplace transform to both sides of the differential equation.
2. Solving the resulting algebraic equation for $F(s)$.
3. Reverse the Laplace transform to find $f(t)$.

Example 1 Consider a first-order ODE $y' + y = e^{-t}$ with initial condition $y(0) = 0$. To solve it, we first apply the Laplace transform to each term:

$$\begin{aligned}\mathcal{L}\{y'(t)\} + \mathcal{L}\{y(t)\} &= \mathcal{L}\{e^{-t}\} \\ sY(s) - y(0) + Y(s) &= \frac{1}{s+1}\end{aligned}$$

Since $y(0) = 0$, this simplifies to:

$$sY(s) + Y(s) = \frac{1}{s+1} \quad \Rightarrow \quad Y(s) = \frac{1}{(s+1)^2}$$

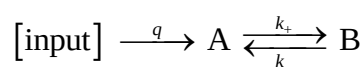
In a table of standard Laplace transforms, we find that $1/(s+1)^2$ corresponds to te^{-t} , in the time domain. Therefore, $y(t) = te^{-t}$.

Example 2 Consider a second-order ODE $f''(t) = -\omega^2 f(t)$ with initial conditions $f(0) = 0$ and $f'(0) = 1$. After applying the Laplace transform to both sides of the equation and working through two rounds of integration by parts, we obtain:

$$1 + s^2 F(s) = -\omega^2 F(s) \quad \Rightarrow \quad F(s) = -\frac{1}{\omega} \frac{\omega}{s^2 + \omega^2}$$

Looking up the reverse transformation yields $f(t) = \omega^{-1} \sin(\omega t)$, which fits both the differential equation and the initial condition.

Example 3 Consider a chemical reaction with a constant source for the reactant



described by the following system of differential equations with zero initial conditions:

$$\begin{cases} \frac{d}{dt} A(t) = -k_+ A(t) + k_- B(t) + q \\ \frac{d}{dt} B(t) = k_+ A(t) - k_- B(t) \end{cases}$$

Calculating the Laplace transform of both sides yields:

$$\begin{cases} sA(s) = -k_+ A(s) + k_- B(s) + q/s \\ sB(s) = k_+ A(s) - k_- B(s) \end{cases}$$

Solving this for the Laplace transforms of the two concentrations yields:

$$\begin{cases} A(s) = \frac{q}{s} \frac{k_+ + s}{k_- + s} \\ B(s) = \frac{k_+}{k_- + s} A(s) \end{cases}$$

where $k_T = k_+ + k_-$. Using partial fraction expansion, we obtain:

$$A(s) = \frac{qk_-}{k_T s^2} + \frac{qk_-}{k_T^2 s} - \frac{k_+}{k_T^2 (k_T + s)}$$

At this point, the Laplace transform is easily inverted:

$$A(t) = \frac{qk_-}{k_T} t + \frac{qk_-}{k_T^2} - \frac{k_+}{k_T^2} e^{-k_T t}$$

The easiest way to obtain $B(t)$ is to use conservation of matter: $A(t) + B(t) = qt$.

4. Workshop questions

1. Calculate the Laplace transform of the following functions:

(a) $f(t) = 3$; (b) $f(t) = t^2$; (c) $f(t) = e^{4t}$; (d) $f(t) = \cos(3t)$; (e) $f(t) = \sin^2(2t)$

2. Use the properties of Laplace transforms to compute the following:

(a) Given $f(t) = 5e^{3t} + 4\cos(2t)$, find $\mathcal{L}\{f(t)\}$.

(b) For $g(t) = t^2 e^{-2t}$, use the differentiation property to find $\mathcal{L}\{g'(t)\}$.

(c) For $h(t) = \theta(t-3)$ where $\theta(t)$ is the Heaviside step function, compute $\mathcal{L}\{h(t)\}$ using the time shift property.

3. Solve the following differential equations using Laplace transforms:

(a) $y'' + 4y = 0$ with initial conditions $y(0) = 1$ and $y'(0) = 0$.

(b) $y'' + 3y' + 2y = e^{-t}$ with initial conditions $y(0) = 0$ and $y'(0) = 1$.

4. Find the inverse Laplace Transform of the following functions:

(a) $F(s) = \frac{6}{s+2}$; (b) $F(s) = \frac{2s+3}{s^2+4s-5}$; (c) $F(s) = \frac{1}{s^2(s-1)}$