

CHEG2002 - Week 21, Lecture 2 - Transfer function models

Reading: Seborg, Edgar, Mellichamp, and Doyle, "Process Dynamics and Control", 3rd ed., Chapter 4.

In the previous lecture, our approach to modelling a physical system was to find the differential equation that describes it and to solve it for some initial condition. This is inconvenient in process engineering where the output of one system may be the input of the next one – the equations describing the entire installation may be overly complicated. It would be more convenient to treat every stage as a separate system with an input and an output:

$$\begin{array}{c} x(t) \end{array} \rightarrow \begin{array}{c} \Phi \end{array} \begin{array}{c} y(t) \end{array} \quad \text{OR} \quad y(t) = \Phi \{x(t)\} \quad (1)$$

and then to chain those stages one after another. In general, this is only possible for *linear time-invariant* (LTI) systems, defined as systems satisfying the following conditions:

$$\begin{aligned} \Phi \{ \alpha x_1(t) + \beta x_2(t) \} &= \alpha \Phi \{ x_1(t) \} + \beta \Phi \{ x_2(t) \} \\ \Phi \{ x(t - t_0) \} &= y(t - t_0) \end{aligned} \quad (2)$$

where $x(t)$ are input signals, $y(t)$ are output signals and α, β, t_0 are constants. In other words, an LTI system behaves *linearly* with respect to the input signal – a sum of two input signals yields a sum of their corresponding outputs, and multiplying an input by a constant multiplies the output by the same constant. If the input is shifted in time by a constant amount, the system output is shifted by the same amount, but otherwise remains the same – this is called *time invariance*.

1. Transfer functions of LTI systems

For a Linear Time-Invariant (LTI) system, the *transfer function* $H(s)$ is defined as the ratio of the Laplace transform of the output signal $Y(s)$ to the Laplace transform of the input signal $X(s)$, assuming zero initial condition on $x(t)$:

$$H(s) = \frac{Y(s)}{X(s)}, \quad x(0) = 0 \quad (3)$$

so that $Y(s) = H(s)X(s)$. This definition makes it easy to deal with chains and trees of systems in which the outputs of preceding systems are inputs of subsequent ones. For example:

$$Y_C(s) = H_C(s)H_B(s)H_A(s)X_A(s) \quad (4)$$

gives the Laplace transform of the output of LTI system C in a chain that starts with the input $x_A(t)$ fed into system A, the output of A fed into system B, and the output of B then fed into system C.

2. Deviation variables

Real-life initial conditions are rarely zero. However, LTI systems allow variable substitutions that use *deviations* from specified reference values. As an example, consider a flow buffer:

$$\rho V \frac{d}{dt} x_{\text{out}}(t) = -w_{\text{out}} x_{\text{out}}(t) + w_{\text{in}} x_{\text{in}}(t) \quad (5)$$

where w_{in} is the flow rate at the inlet with concentration $x_{in}(t)$ and w_{out} is the flow rate at the outlet with concentration $x_{out}(t)$. The flow rates are assumed to be time-independent. The Laplace transform contains an inconvenient initial condition:

$$\rho V [sX_{out}(s) - x_{out}(0)] = -w_{out}X_{out}(s) + w_{in}X_{in}(s) \quad (6)$$

that prevents the Laplace domain relation from being cast in the form prescribed by Eq (3). The reason is that the initial condition $x_{out}(0)$ is not zero. This problem may be avoided by assuming that the system has started its evolution in a steady state, such that:

$$0 = -w_{out}\bar{x}_{out} + w_{in}\bar{x}_{in} \quad (7)$$

where overbar denotes constant reference values that satisfy this steady state equation. We then subtract this equation from Eq (5) term by term to obtain:

$$\rho V \frac{d}{dt} x_{out}(t) = -w_{out} [x_{out}(t) - \bar{x}_{out}] + w_{in} [x_{in}(t) - \bar{x}_{in}] \quad (8)$$

and also note that subtracting a constant under the derivative does not change the equation:

$$\rho V \frac{d}{dt} [x_{out}(t) - \bar{x}_{out}] = -w_{out} [x_{out}(t) - \bar{x}_{out}] + w_{in} [x_{in}(t) - \bar{x}_{in}] \quad (9)$$

A variable substitution then yields a differential equation with a zero initial condition:

$$\begin{cases} y_{out}(t) = x_{out}(t) - \bar{x}_{out} \\ y_{in}(t) = x_{in}(t) - \bar{x}_{in} \end{cases} \Rightarrow \rho V \frac{d}{dt} y_{out}(t) = -w_{out}y_{out}(t) + w_{in}y_{in}(t) \quad (10)$$

which may now be Laplace transformed in a way that yields the transfer function:

$$\rho V s Y_{out}(s) = -w_{out}Y_{out}(s) + w_{in}Y_{in}(s) \Rightarrow Y_{out}(s) = \frac{w_{in}}{\rho V s + w_{out}} Y_{in}(s) \quad (11)$$

After introducing the flow ratio $k = w_{in}/w_{out}$ and the characteristic outflow time $\tau = \rho V/w_{out}$, the transfer function acquires a particularly simple form:

$$H(s) = \frac{k}{\tau s + 1} \quad (12)$$

3. Multiple inputs and outputs

Let us upgrade our flow buffer and make it a blending tank with two inputs:

$$\rho V \frac{d}{dt} x_{out}(t) = -w_{out}x_{out}(t) + w_{in}^{(1)}x_{in}^{(1)}(t) + w_{in}^{(2)}x_{in}^{(2)}(t) \quad (13)$$

A similar steady state condition for the reference values:

$$0 = -w_{out}\bar{x}_{out} + w_{in}^{(1)}\bar{x}_{in}^{(1)} + w_{in}^{(2)}\bar{x}_{in}^{(2)} \quad (14)$$

and an identical set of subsequent mathematical transformations yields:

$$Y_{out}(s) = \frac{k_1}{\tau s + 1} Y_{in}^{(1)}(s) + \frac{k_2}{\tau s + 1} Y_{in}^{(2)}(s), \quad k_{1,2} = w_{in}^{(1,2)}/w_{out} \quad (15)$$

where we can see that each input has its own transfer function, and that inputs are additive. A similar treatment for a system with multiple inputs and multiple outputs reveals that there is a separate transfer function for each input-output pair.

4. Linearisation of non-linear models

For a general equation of motion

$$\frac{dx_{\text{out}}}{dt} = f \left[x_{\text{out}}(t), x_{\text{in}}^{(1)}(t), x_{\text{in}}^{(2)}(t), \dots \right] \quad (16)$$

involving multiple inputs and some differentiable non-linear function f , the transfer function approach works only in the *limit of small deviations* from equilibrium values $\bar{x}_k$ and $\bar{y}$ which must exist. The deviations must be small enough for the Taylor series:

$$f(x_{\text{out}}, x_{\text{in}}^{(1)}, \dots) = f(\bar{x}_{\text{out}}, \bar{x}_{\text{in}}^{(1)}, \dots) + \left(\frac{\partial f}{\partial x_{\text{out}}} \right) (x_{\text{out}} - \bar{x}_{\text{out}}) + \left(\frac{\partial f}{\partial x_{\text{in}}^{(1)}} \right) (x_{\text{in}}^{(1)} - \bar{x}_{\text{in}}^{(1)}) + \dots \quad (17)$$

to be a sufficiently good approximation at first order. In this series, the derivatives are calculated at $(\bar{x}_{\text{out}}, \bar{x}_{\text{in}}^{(1)}, \dots)$, and therefore are constants. As an example, consider the flow buffer from Section 2, but now with a time-dependent inflow rate $w_{\text{in}}(t)$ that makes the equation non-linear:

$$\rho V \frac{d}{dt} x_{\text{out}}(t) = -w_{\text{out}} x_{\text{out}}(t) + w_{\text{in}}(t) x_{\text{in}}(t) \quad (18)$$

With all of that in place, our function on the right hand side is:

$$f(x_{\text{out}}, x_{\text{in}}, w_{\text{in}}) = -w_{\text{out}} x_{\text{out}} + w_{\text{in}} x_{\text{in}} \quad (19)$$

and the Taylor series therefore becomes:

$$f(x_{\text{out}}, x_{\text{in}}, w_{\text{in}}) = f(\bar{x}_{\text{out}}, \bar{x}_{\text{in}}, \bar{w}_{\text{in}}) + \left(\frac{\partial f}{\partial x_{\text{out}}} \right) (x_{\text{out}} - \bar{x}_{\text{out}}) + \left(\frac{\partial f}{\partial x_{\text{in}}} \right) (x_{\text{in}} - \bar{x}_{\text{in}}) + \left(\frac{\partial f}{\partial w_{\text{in}}} \right) (w_{\text{in}} - \bar{w}_{\text{in}}) + \dots \quad (20)$$

where the three derivatives are evaluated at $(\bar{x}_{\text{out}}, \bar{x}_{\text{in}}, \bar{w}_{\text{in}})$. A simple exercise in calculus yields:

$$\rho V \frac{dx_{\text{out}}}{dt} = -w_{\text{out}} x_{\text{out}} + \bar{w}_{\text{in}} x_{\text{in}} + \bar{x}_{\text{in}} (w_{\text{in}} - \bar{w}_{\text{in}}) \quad (21)$$

Introducing deviation variables for x_{in} and x_{out} in the same way as it was done in Section 2 yields:

$$\rho V \frac{dy_{\text{out}}}{dt} = -w_{\text{out}} y_{\text{out}} + \bar{w}_{\text{in}} y_{\text{in}} + \bar{x}_{\text{in}} u_{\text{in}}, \quad u_{\text{in}} = w_{\text{in}} - \bar{w}_{\text{in}} \quad (22)$$

where u_{in} is a deviation variable for the flow rate. At this point, the transfer relationship in the Laplace domain is easily obtained:

$$Y_{\text{out}}(s) = \frac{w_{\text{in}}}{\rho V s + w_{\text{out}}} Y_{\text{in}}(s) + \frac{\bar{x}_{\text{in}}}{\rho V s + w_{\text{out}}} U_{\text{in}}(s) \quad (23)$$

where there are now two inputs (the inflow concentration and the inflow rate) connected to the output (outflow concentration) by their corresponding transfer functions.

5. Pulse response of an LTI system

A short and sharp input signal can be modelled by *Dirac's delta-function* which is a mathematical representation of an infinitely sharp peak with unit area:

$$\delta(t) = \begin{cases} 0 & t \neq 0 \\ \infty & t = 0 \end{cases} \quad \int_{-\infty}^{\infty} \delta(t) dt = 1 \quad (24)$$

The delta-function has the following property:

$$x(t) = \int_{-\infty}^{\infty} x(\tau) \delta(\tau - t) d\tau \quad (25)$$

Let us apply our LTI system to both sides of this expression. The system Φ only acts on functions of physical time t , and sees the integration variable τ as a constant, therefore:

$$\Phi\{x(t)\} = \Phi\left\{\int_{-\infty}^{\infty} x(\tau) \delta(\tau - t) d\tau\right\} = \int_{-\infty}^{\infty} x(\tau) \Phi\{\delta(\tau - t)\} d\tau = \int_{-\infty}^{\infty} x(\tau) h(\tau - t) d\tau \quad (26)$$

The function $h(t) = \Phi\{\delta(t)\}$ is called the *pulse response* of the system. Eq (26) indicates that it allows us to predict system response to an arbitrary input $x(t)$.

6. Workshop questions

1. For the processes modelled by the following system of equations for deviation variables $f(t)$, $g(t)$, $u(t)$, $v(t)$:

$$\begin{cases} \frac{d}{dt} f(t) = -2f(t) - 3g(t) + 2u(t) \\ \frac{d}{dt} g(t) = 2f(t) - 3g(t) + u(t) + v(t) \end{cases}$$

find the four transfer functions relating the outputs $f(t)$ and $g(t)$ to the inputs $u(t)$ and $v(t)$. Under what conditions would this dynamical system have a fixed steady state?

2. Find first-order Taylor expansions for the following functions:

(a) $f(x, y) = x \sin(xy)$ around $(x_0, y_0) = (0, 0)$;

(b) $f(x, y) = xy^2 + x^2y$ around $(x_0, y_0) = (1, 1)$;

(c) $f(x, y, z) = e^{x^2+y^2+z^2} + x + y + z$ around $(x_0, y_0, z_0) = (0, 0, 0)$.

3. Perform the deviation variable transformation in Eq (21) to obtain Eq (22) and hence, after the Laplace transform, Eq (23).