

CHEG2002 - Lecture 3 - Dynamic behaviour of LTI systems, part 1

Reading: Seborg, Edgar, Mellichamp, and Doyle, "Process Dynamics and Control", 3rd ed., Chapter 5.

In this lecture we illustrate the Laplace transform formalism by looking at common calculation types – initial states, steady states, and responses to simple inputs: step, ramp, and sine wave. In all cases, Laplace transform methods yield useful and easily interpretable analytical solutions.

1. Final and initial value theorems

Consider a system with an input $x(t)$ for which the output limit $y(t \rightarrow \infty)$ exists. A convenient way of calculating that limit is to use the Laplace transform:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} [sY(s)] \quad (1)$$

The proof is by using the derivative property $\mathcal{L}\{y'(t)\} = sY(s) - y(0)$:

$$\begin{aligned} \lim_{s \rightarrow 0} [sY(s)] &= \lim_{s \rightarrow 0} [\mathcal{L}\{y'(t)\} + y(0)] = \lim_{s \rightarrow 0} \left[\int_0^{\infty} y'(t) e^{-st} dt + y(0) \right] = \\ &= \int_0^{\infty} y'(t) \lim_{s \rightarrow 0} [e^{-st}] dt + y(0) = \int_0^{\infty} y'(t) dt + y(0) = \lim_{t \rightarrow \infty} y(t) \end{aligned} \quad (2)$$

This is *final value theorem*. A similar statement with a similar proof is the *initial value theorem*:

$$\lim_{t \rightarrow 0} y(t) = \lim_{s \rightarrow \infty} [sY(s)] \quad (3)$$

Both theorems require the corresponding limits of $y(t)$ to exist. For the final state theorem, this creates an *applicability condition* that there must be no $e^{\lambda t}$ terms in $y(t)$ with λ in the right half-plane of the complex plane because those correspond to rising exponentials and undamped sinusoidal oscillations that have no $t \rightarrow \infty$ limit. In the Laplace domain that means that $Y(s)$ must have no poles in the right half-plane of the complex plane.

Example 1: consider a system described by the differential equation

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = u(t)$$

where $u(t)$ is a unit step function and the initial conditions are $y'(0) = y(0) = 0$. Find the asymptotic value $y(t \rightarrow \infty)$.

Solution: we transform the equation into the Laplace domain using derivative properties and the standard result for the step function

$$s^2 Y(s) + 3sY(s) + 2Y(s) = \frac{1}{s}$$

After solving this for $Y(s)$, we obtain:

$$Y(s) = \frac{1}{s(s^2 + 3s + 2)}$$

The limit in the final value theorem is then easy to evaluate:

$$\lim_{t \rightarrow \infty} y(t) = \lim_{s \rightarrow 0} [sY(s)] = \lim_{s \rightarrow 0} \frac{1}{s^2 + 3s + 2} = \frac{1}{2}$$

The same result may be obtained in the time domain by noting that, at the steady state, all derivatives are zero and the step function is equal to 1.

2. Standard input library

The following standard inputs occur frequently in practical calculations. We have already obtained their Laplace transforms in the previous lectures:

(a) *Step* of amplitude A :

$$x(t) = \begin{cases} 0 & t < 0 \\ A & t \geq 0 \end{cases} \quad X(s) = \frac{A}{s} \quad (4)$$

(b) *Ramp* with slope a :

$$x(t) = \begin{cases} 0 & t < 0 \\ at & t \geq 0 \end{cases} \quad X(s) = \frac{a}{s^2} \quad (5)$$

(c) *Square pulse* of amplitude A and width w :

$$x(t) = \begin{cases} 0 & t < 0 \\ A & 0 \leq t < w \\ 0 & t \geq w \end{cases} \quad X(s) = \frac{A}{s}(1 - e^{-ws}) \quad (6)$$

(d) *Monochromatic waves* of amplitude A and frequency ω :

$$\begin{aligned} x(t) &= \begin{cases} 0 & t < 0 \\ A \sin(\omega t) & t \geq 0 \end{cases} & X(s) &= \frac{A\omega}{s^2 + \omega^2} \\ x(t) &= \begin{cases} 0 & t < 0 \\ A \cos(\omega t) & t \geq 0 \end{cases} & X(s) &= \frac{As}{s^2 + \omega^2} \end{aligned} \quad (7)$$

3. Response functions of first-order processes

A first-order system with an amplification coefficient k and a time constant τ has the following transfer relation in the Laplace domain:

$$Y(s) = \frac{k}{\tau s + 1} X(s) \quad (8)$$

(a) The response to the step input with amplitude A :

$$Y(s) = \frac{k}{\tau s + 1} \frac{A}{s} = \frac{kA}{s} - \frac{kA\tau}{1 + \tau s} \quad \Rightarrow \quad y(t) = kA(1 - e^{-t/\tau}) \quad (9)$$

multiplies the amplitude of the input by k and converges to the updated amplitude kA exponentially with a time constant τ .

(b) The response to the ramp input with slope a :

$$Y(s) = \frac{k}{\tau s + 1} \frac{a}{s^2} = \frac{k a \tau^2}{\tau s + 1} - \frac{k a \tau}{s} + \frac{k a}{s^2} \Rightarrow y(t) = k a \tau (e^{-t/\tau} - 1) + k a t \quad (10)$$

As expected, the output depends linearly on the input amplitude (as it should be) and the amplification coefficient of the system itself. The exponential vanishes eventually, and the asymptotic solution $y(t \gg \tau) = k a (t - \tau)$ is essentially the input, scaled by the amplification coefficient k and delayed by the time constant τ .

- (c) The pattern wherein the asymptotic response is a scaled and delayed version of the input continues with the sinusoidal input with amplitude A and frequency ω :

$$Y(s) = \frac{k}{\tau s + 1} \frac{A \omega}{s^2 + \omega^2} \Rightarrow y(t) = \frac{k A \omega \tau}{\omega^2 \tau^2 + 1} e^{-t/\tau} + \frac{k A}{\sqrt{\omega^2 \tau^2 + 1}} \sin(\omega t - \varphi) \quad (11)$$

where the delay $\varphi = \arctan(\omega \tau)$ is now a function of both the response time and the input frequency. The first term on the right hand side eventually fades.

4. Workshop questions

1. A process with an input $x(t)$ and an output $y(t)$ has the transfer function:

$$\frac{Y(s)}{X(s)} = \frac{2}{s^2 + 2s + 1}$$

- (a) For a step change in the input with $X(s) = 2/s$, find the time-domain response $y(t)$.
- (b) For the same input, obtain the steady state value of $y(t)$ using:
- (i) final value theorem;
 - (ii) $t \rightarrow \infty$ limit of the time-domain solution;
 - (iii) steady-state limit of the time-domain equation of motion.
2. For the process in Problem 1, explain why the final state theorem yields nonsensical results when the input is $x(t) = \sin(2t)$. What is the correct asymptotic solution $y(t \gg 1)$?
3. For the process in Problem 1, find the time-domain response to a pulse input $x(t) = \delta(t - \varepsilon)$, where $\delta(t)$ is a delta-function and ε is a small positive number for which the limit $\varepsilon \rightarrow 0$ should be taken after the Laplace transform.