

CHEG2002 - Lecture 4 - Dynamic behaviour of LTI systems, part 2

Reading: Seborg, Edgar, Mellichamp, and Doyle, "Process Dynamics and Control", 3rd ed., Chapter 5.

In this lecture we continue looking at the applications of the Laplace transform formalism to LTI systems. Responses of second-order processes to common inputs are covered.

1. Series RLC circuit

A physical system that provides some intuition for the exploration of second order processes is a series RLC circuit (Figure 1): sequentially connected *resistor*, *inductor*, *capacitor*, and a *voltage source*.

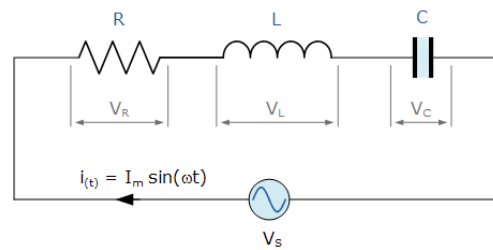


Figure 1. Series RLC circuit, containing a voltage source, a resistance, an inductor, and a capacitor.

We start with *Kirchhoff's voltage law*, which requires voltage integrals over close loops to be zero. In this case, it leads to the following balance of voltages:

$$V_s(t) = V_R(t) + V_L(t) + V_C(t) \quad (1)$$

We assume that only the resistor has a non-zero resistance and use *Ohm's law*:

$$V_R(t) = RI(t) \quad (2)$$

From the *inductor equation* (inductor voltage is proportional to rate of change in current):

$$V_L(t) = LI'(t) \quad (3)$$

From the *capacitor equation* (capacitor current is proportional to the rate of change in voltage):

$$I(t) = CV'_C(t) \quad (4)$$

We also know the relationship between charge and current:

$$I(t) = \frac{dq}{dt} \quad (5)$$

Combining all of these, we get the following equation for the charge on the capacitor:

$$\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q(t) = \frac{1}{L} V_s(t) \quad (6)$$

where q is the charge on the capacitor, V_s is source voltage, R is the resistance of the resistor, C is the capacitance of the capacitor, and L is the inductance of the inductor.

2. Responses of second-order processes

When the initial charge on the capacitor is zero, the Laplace transform of Eq (6) is:

$$s^2 q(s) + \frac{R}{L} s q(s) + \frac{1}{LC} q(s) = \frac{1}{L} V_s(s) \quad (7)$$

It may be rearranged into the canonical form:

$$Q(s) = \frac{C}{\tau^2 s^2 + 2\zeta\tau s + 1} V_s(s) \quad \tau = \sqrt{LC} \quad \zeta = \frac{R}{2} \sqrt{\frac{C}{L}} \quad (8)$$

where τ is the circuit **response time**, ζ is the **damping ratio**, and the multiplicative coefficient connecting voltage and charge is C as befits a capacitor. The roots of the transfer function denominator determine the functional type of the inverse Laplace transform (exponential or trigonometric) and therefore the type of response the circuit shows to the voltage step. Based on the discriminant of the quadratic equation $4\zeta^2\tau^2 - 4\tau^2(\zeta^2 - 1)$, the following patterns exist:

- (a) **Overdamped**: $\zeta > 1$, yielding two unequal real roots. The response to a voltage step from an initial zero voltage to some finite voltage V_{in}

$$q(s) = \frac{C}{\tau^2 s^2 + 2\zeta\tau s + 1} \frac{V_{in}}{s} \quad (9)$$

shows an exponential cross-fade from zero to $q = CV_{in}$:

$$q(t) = CV_{in} \left(1 - \frac{\tau_1 e^{-t/\tau_1} - \tau_2 e^{-t/\tau_2}}{\tau_1 - \tau_2} \right) \quad \tau_{1,2} = \frac{\tau}{\zeta \mp \sqrt{\zeta^2 - 1}} \quad (10)$$

with the time constants depending on the damping parameter ζ . From the physical perspective, in this regime the capacitor responds to a change in the input voltage by simply charging or draining *via* an exponential cross-fade (Figure 2).

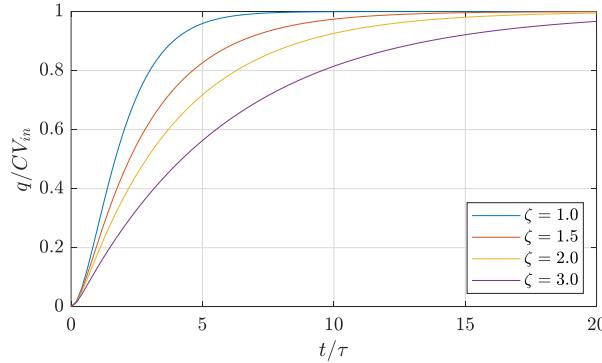


Figure 2. Charge on the capacitor as a function of time in a critically damped ($\zeta = 1.0$) and overdamped ($\zeta > 1.0$) RLC circuit for different values of the damping parameter.

- (b) **Critically damped**: $\zeta = 1$, yielding two equal real roots. The response to the step input is still an exponential cross-fade, but with only one time constant (Figure 2, blue curve):

$$q(t) = CV_{in} \left(1 - \left(1 + \frac{t}{\tau} \right) e^{-t/\tau} \right) \quad (11)$$

From the physical perspective, this corresponds to the fastest response of the capacitor charge to the input voltage that still keeps the circuit in the exponential response regime.

(c) **Underdamped:** $0 < \zeta < 1$, yielding two conjugate complex roots and a fading oscillation:

$$q(t) = CV_{in} \left\{ 1 - e^{-\zeta t/\tau} \left[\cos\left(\frac{\sqrt{1-\zeta^2}t}{\tau}\right) + \frac{\zeta}{\sqrt{1-\zeta^2}} \sin\left(\frac{\sqrt{1-\zeta^2}t}{\tau}\right) \right] \right\} \quad (12)$$

The response is now faster (Figure 3) compared to the cases with stronger damping (Figure 2), but it also became oscillatory because the inductor is now strong enough to keep charging the capacitor beyond its steady state.

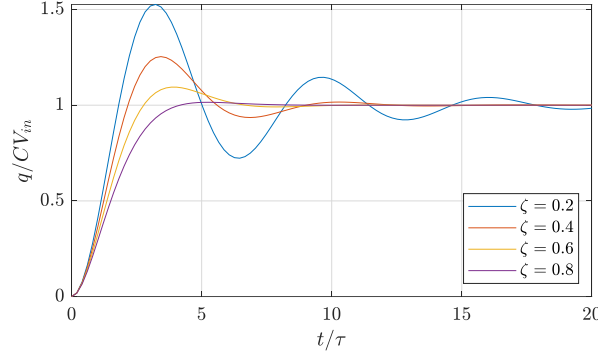


Figure 3. Charge on the capacitor as a function of time in an underdamped RLC circuit for different values of the damping parameter.

The following parameters of underdamped second-order systems are pertinent:

- (i) **rise time:** the time to first reach $y_{\infty} = CV_{in}$.
- (ii) **time to first peak:** $t_p = \pi\tau / \sqrt{1-\zeta^2}$.
- (iii) **settling time:** time to reach and stay inside $y_{\infty} \pm 5\%$.
- (iv) **overshoot:** the ratio of the additional value at the maximum to the steady state value

$$\frac{y_{\max} - y_{\infty}}{y_{\infty}} = \exp\left(-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}\right) \quad (13)$$

- (v) **decay ratio:** the ratio of the second (in time order) maximum to the first maximum

$$\frac{y_{\max}^{(2)}}{y_{\max}^{(1)}} = \exp\left(-\frac{2\pi\zeta}{\sqrt{1-\zeta^2}}\right) \quad (14)$$

- (vi) **oscillation period:** the period of the trigonometric functions in Eq (12).

$$P = \frac{2\pi\tau}{\sqrt{1-\zeta^2}} \quad (15)$$

3. Progress test questions

1. For each of the following functions, find the Laplace transform:

(a) $y(t) = te^{-at}$

(b) $y(t) = t \sin(\omega t)$

(c) $y(t) = te^{-at} \sin(\omega t)$

2. The Laplace transform of a function $y(t)$ is $Y(s) = 1/(s^2 + 15s)$. Find $y(t)$.

3. Consider the system defined by:

$$y(t) = tx(t+1)$$

(a) Is this system linear? Why or why not? Justify your answer.

(b) Is this system time-invariant? Why or why not? Justify your answer.

(c) Is this system stable? Why or why not? Justify your answer.

(d) Is this system causal? Why or why not? Justify your answer.

4. A linear time-invariant system has the following transfer function:

$$H(s) = \frac{2s-1}{(s+3)(s+5)}$$

(a) Determine the differential equation relating the input $x(t)$ to the output $y(t)$.

(b) Is this system stable and why?

(c) Find the output when the input is $x(t) = e^{-3t}u(t)$ where $u(t)$ is the step function.