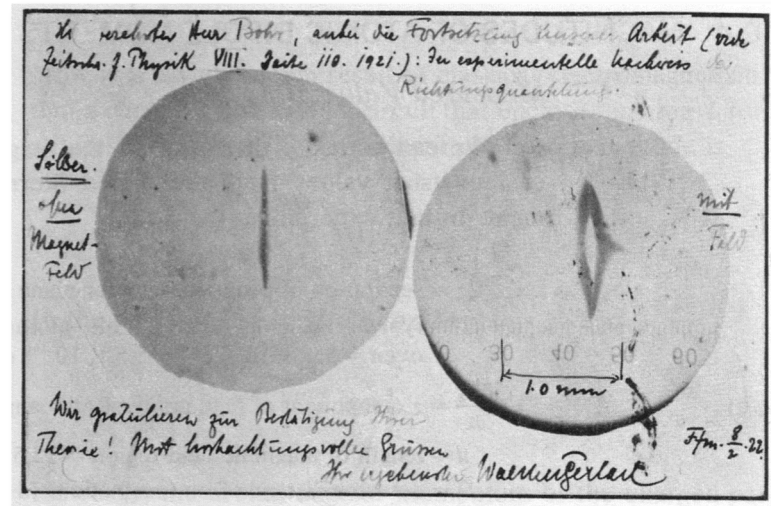
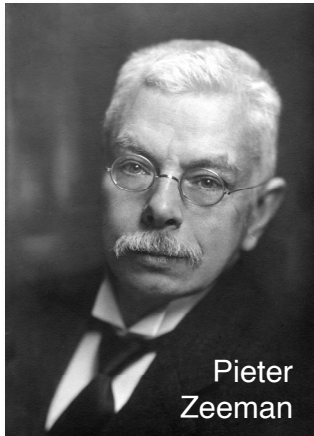




Spin as a fundamental physical property

some history, some conservation law mathematics, and a sketch of where exactly spin comes from

1896-1922: the tale of two Zeeman effects

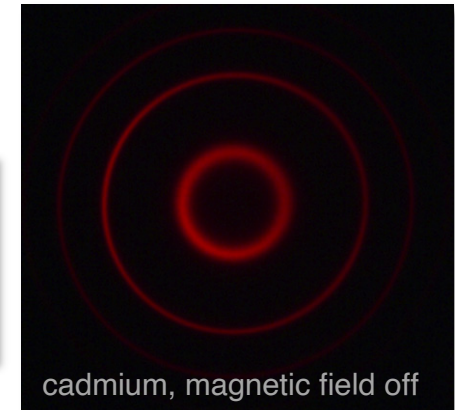


Pieter Zeeman

Zeeman, in 1896, in *Verslagen van de Gewone Vergaderingen der Wis- en Natuurkundige Afdeeling der Koninklijke Akademie van Wetenschappen te Amsterdam*:

4. De BUNSEN-vlam werd vervangen door een gas-zuurstofvlam. Evenals in 3 werd asbest met keukenzout doortrokken in de vlam gebracht. Deze steeg verticaal tusschen de polen naar boven. Werd nu de stroom weer aangezet dan werden wederom de *D*-lijnen verbreed, zoodat ze wel 3 à 4 malen hun oorspronkelijke breedte kregen.

...followed by quantitative splitting measurements in 1897.



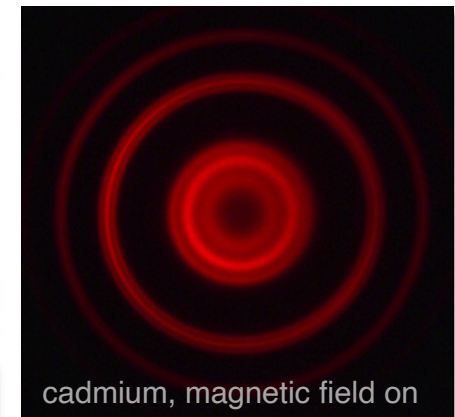
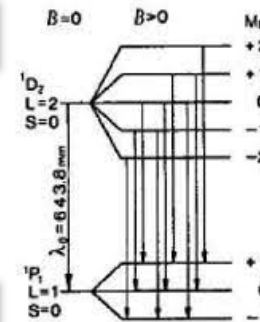
cadmium, magnetic field off

A partial explanation was settled by 1916:

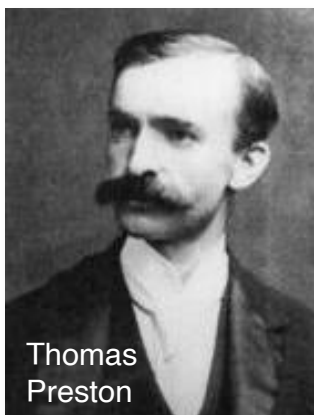
$$\boldsymbol{\mu} = -\frac{e}{2m_e} \mathbf{L}, \quad E = -\boldsymbol{\mu} \cdot \mathbf{B}$$

← orbital angular momentum

spectroscopic quantum numbers n, l, m (Bohr, Sommerfeld, Debye)

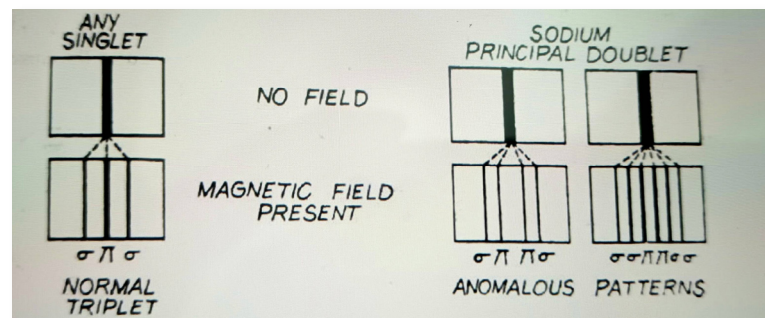


cadmium, magnetic field on



Thomas Preston

But there was a problem (1897):



Wolfgang Pauli

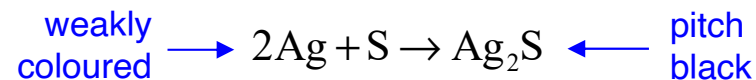
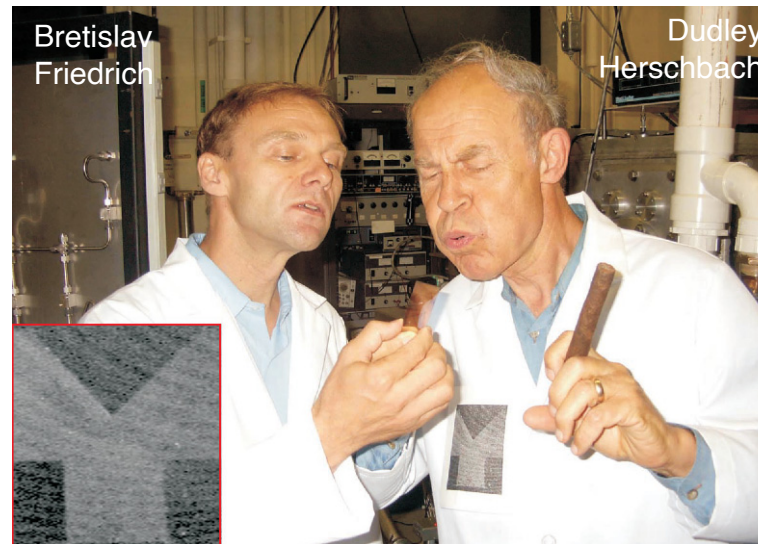
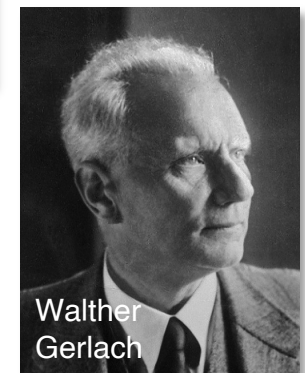
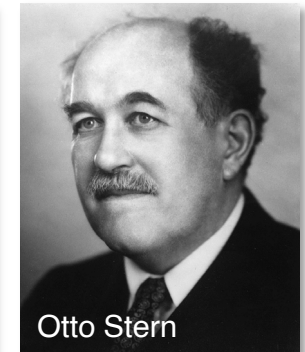
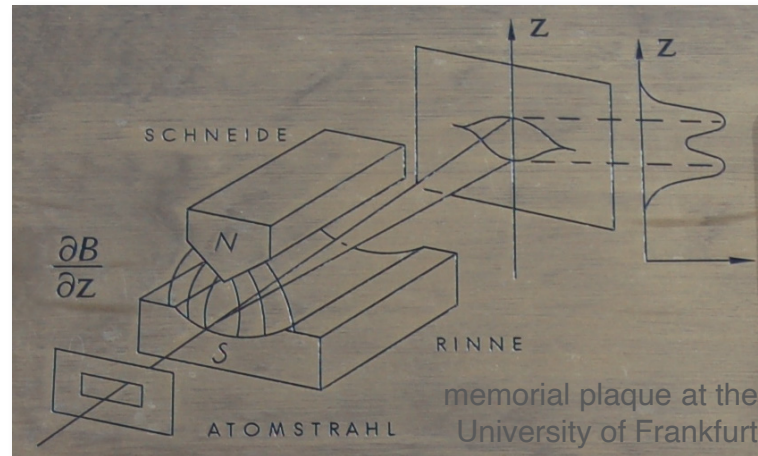
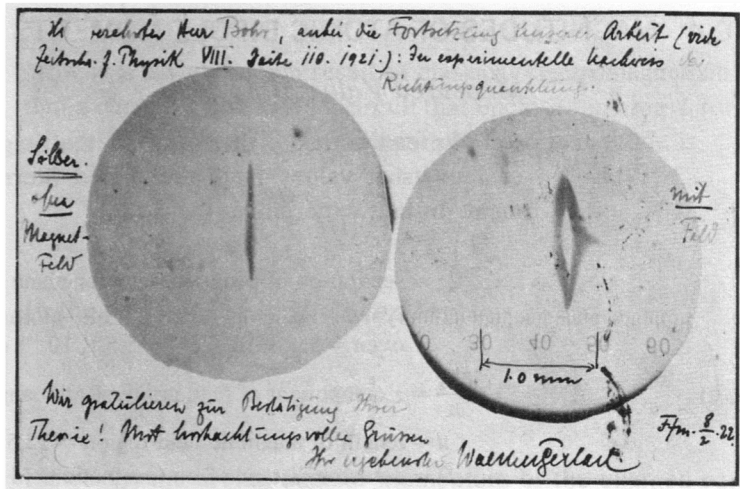
“Wie kann man glücklich aussehen, wenn man an den anomalen Zeeman-effekt denkt?”

Wolfgang Pauli, in 1922

► N.B. Schrödinger solved his equation for the hydrogen atom in 1926.

1922: not-quite-spots, not-yet-spin, and a bad cigar

Wenn sich dieser Unsinn von Bohr als richtig erweisen sollte, geben wir die Physik auf! - Stern and von Laue, 1913



1924-1925: “classically indescribable”

Pauli needed “zweideutigkeit” to reconcile high- and low-field splittings (1925):

HIGH FIELD $(n, l, m_l, m_s): m_l = -l, \dots, +l, m_s = \pm 1/2 \Rightarrow (2l+1) \times 2 = 2(2l+1)$ levels

LOW FIELD $(n, l, j, m_j): j = l \pm 1/2, m_j = -j, \dots, +j \Rightarrow (2l+2) + (2l) = 2(2l+1)$ levels

ADDITIONALLY:

- closed shells known to have no net magnetic moment
- multiplet patterns suggest that each one-electron orbital occurs exactly once

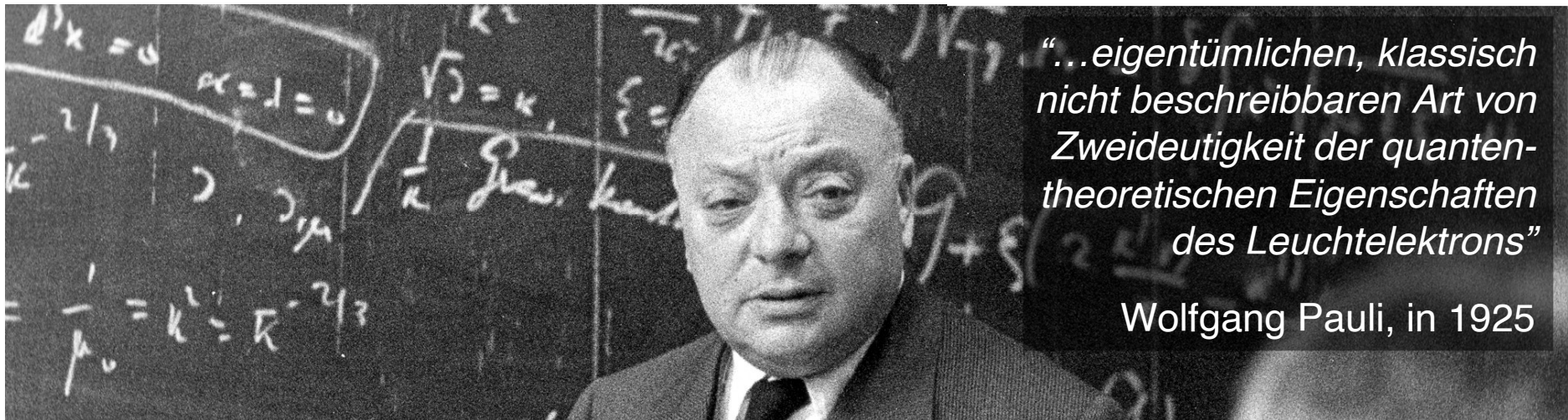


no two
electrons can
have the same
four quantum
numbers

Much later, from group theory:

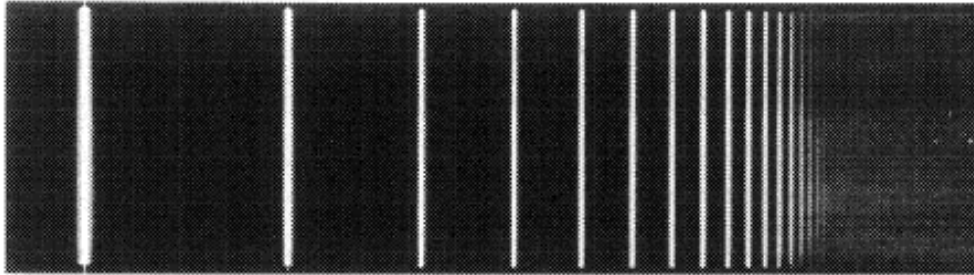
$$\Psi(\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N) = \pm \Psi(\hat{P}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_N\})$$

(the minus comes from QFT)



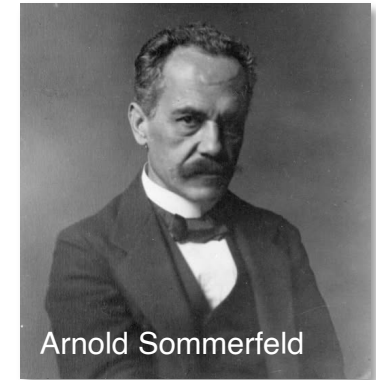
► As of 1925, it was still unclear to Pauli what the “zweideutigkeit” was.

1925: two students and a spectrum



Ballmer (blue-violet) series in the hydrogen atom emission spectrum (10.1103/PhysRev.124.1866).

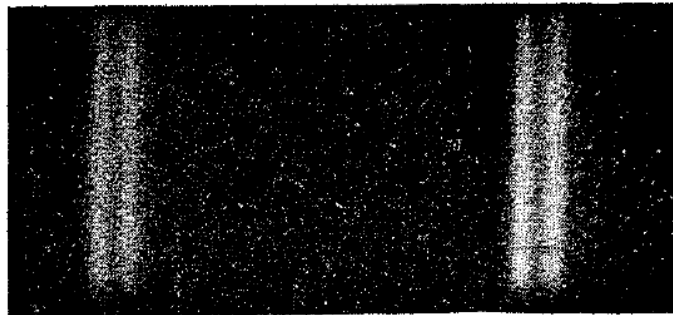
"Spectralzoologie!"
- Wolfgang Pauli



Arnold Sommerfeld



Uhlenbeck



H^1_{α}

H^2_{α}



Goudsmit

Deuterium

Hydrogen

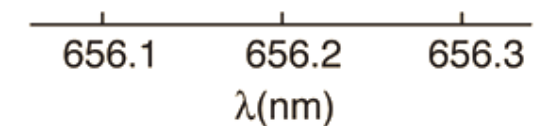
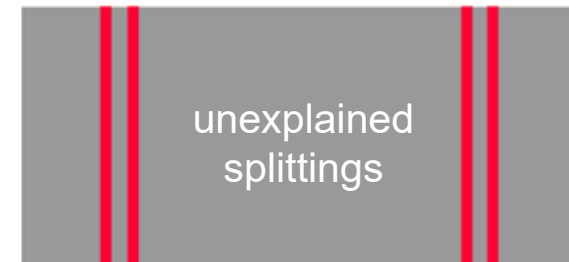


FIG. 9.1.—Photographs of the H_{α} line of both of the hydrogen isotopes H^1 and H^2 . (After Lewis and Spedding.)

Relevant recent circumstances:

- Rutherford-Bohr model of the atom (1913)
- Theory of relativity (special: 1907, general: 1915)
- Relativistic effects in astronomy: ongoing work



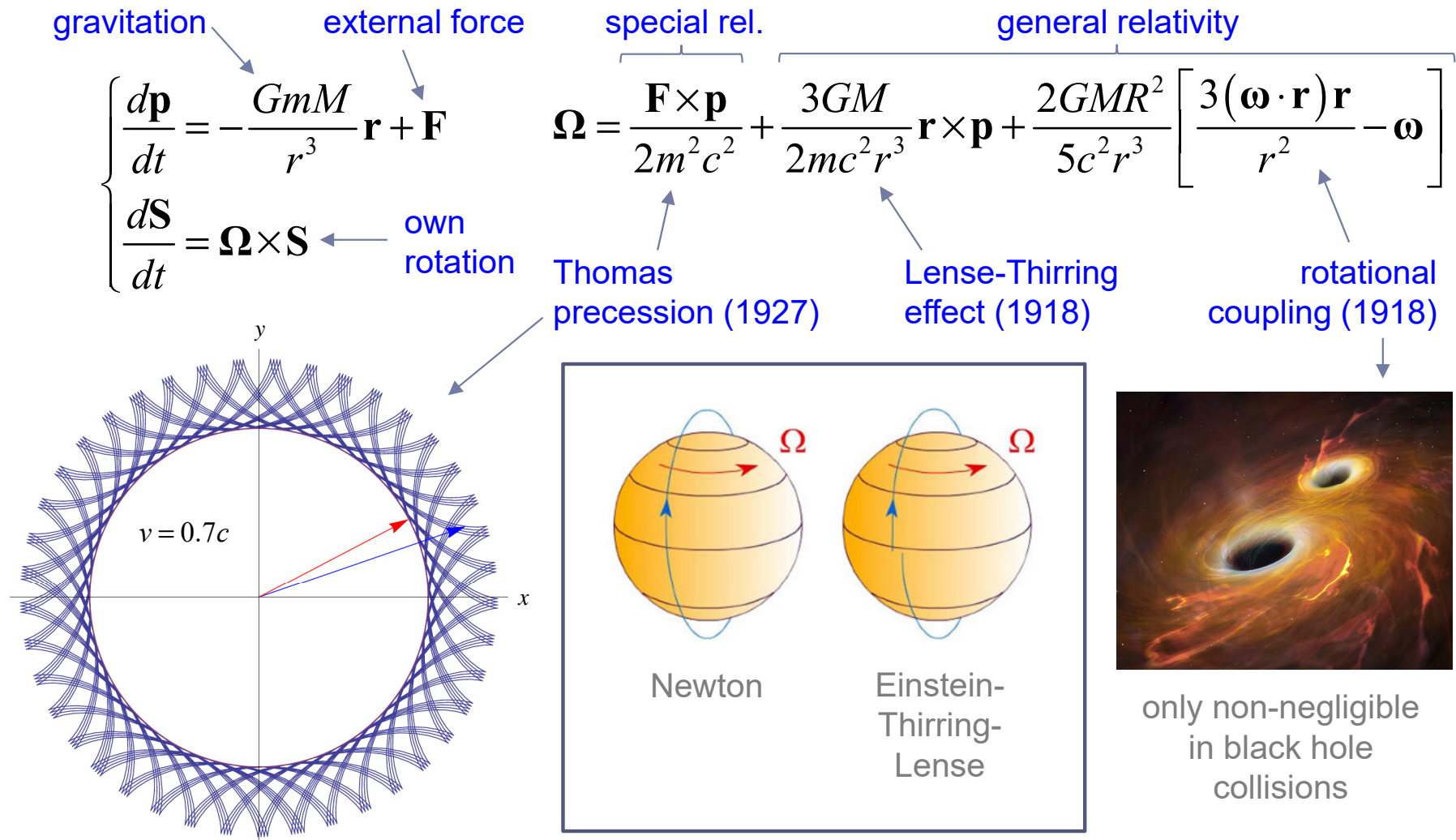
Ehrenfest's reaction:

"Das ist entweder Unsinn
oder etwas sehr Wichtiges."

► What would a pair of "completely clueless" (quote from Sam Goudsmit) PhD students do?

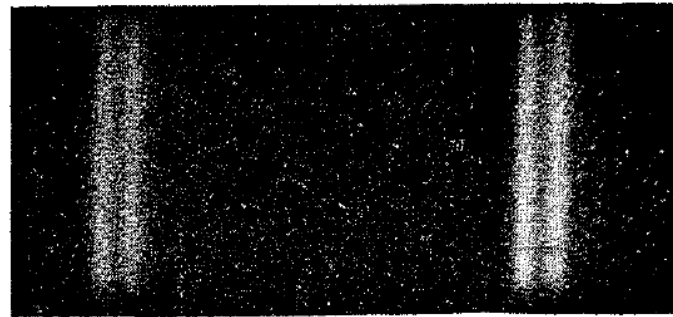
Interlude: macroscopic bodies in astrophysics

Astronomical bodies have orbital, but also intrinsic, angular momenta:



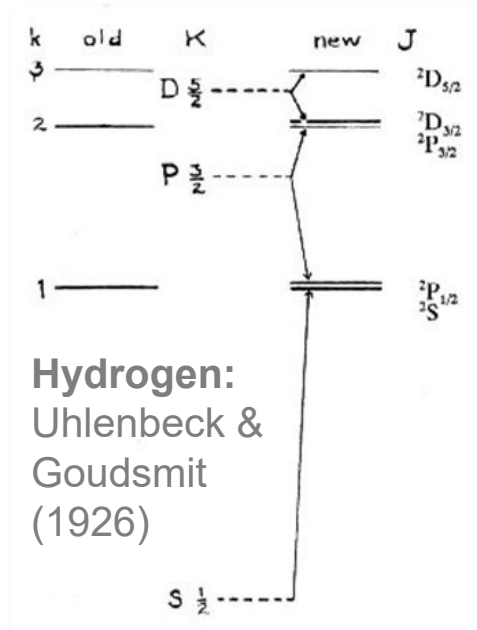
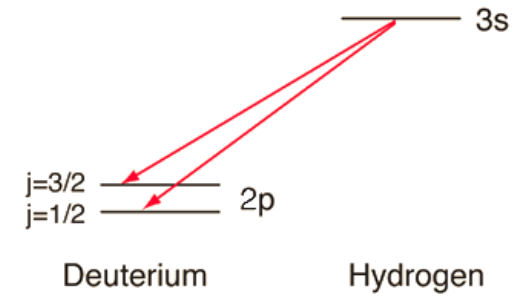
► These are all gravitational effects! Paper timing: Stern-Gerlach (1922); Goudsmit-Uhlenbeck (1925).

1925: two students and a spectrum



H^1_{α}

H^2_{α}



A simple hack:

$$\mathbf{H} \rightarrow \mathbf{H} + \xi \vec{\mathbf{L}} \cdot \vec{\mathbf{S}}$$

“maybe some kind of intrinsic angular momentum”

- Exactly two eigenvalues: not possible. (Pauli)
- Surface velocity: ten times the speed of light. (Lorentz)
- Cannot be angular momentum: not $2j+1$ for integer j .
- Bizarre interpretation in 3D: 4π periodic rotations.
- Heisenberg: “...Seine mutige Note...”, “Wie Sie den Faktor 2 losgeworden sind?”



► Uhlenbeck: “I must say [...] Sam and I [...] had not really appreciated [this] basic difficulty.”

1925-1926: the Great Confusion

Llewellyn
Thomas to
Sam Goudsmit

Universitets Institut
for Teoretisk Fysik
Copenhagen
Denmark
25 March 1926.

Dear Goudsmit

Many thanks for your letter which I was happy to receive this morning. I hope you will be able to understand my English as well as I yours! You have asked me many questions and told me nothing about yourself.

I think you and Uhlenbeck have been very lucky to get your spinning electron published and talked about before Pauli heard of it. It appears that more than a year ago Kronig believed in the spinning electron and worked out something; the first person he showed it to was Pauli. Pauli ridiculed the whole thing so

much that the first person became also the last and no one else heard anything of it. Which all goes to show that the infallibility of the Deity does not extend to his self-styled vicar on earth.

I have inquired about the reprints of your Nature note. None have yet been sent to anyone. Professor Bohr thinks it will be better to wait another ten days to get reprints of my Nature letter to send with yours. There has unfortunately been delay apparently through proofs being lost in the post and my letter cannot now possibly be published before April 3rd. If you think your reprints should be sent off at once please let us know by a post-card.

At present I am still working on a longer article on the kinematics of the rotating electron but hope soon to finish this when I must return to α -particles.

Sugars is still drawing curves of atomic fields and calculating about

Lorentz: "...that is very difficult because it causes the self-energy of the electron to be wrong."

Ehrenfest: "Wel, dat is een aardig idee, al kan het verkeerd zijn. Maar u hebt nog geen reputatie, dus u hebt niets te verliezen."

Bohr: "On your way home you should stop off at Hamburg and explain the factor of 2 to Pauli."

Pauli: "**Eine neue Kopenhagener Irrlehre!**"

▶ Until Thomas found the factor of 2, everyone was intensely uncomfortable with the situation.

1926: the Thomas half

How to deal with a missing factor of 2? Easy:

Elektronenmodells abhängen. Um mit den Tatsachen in Übereinstimmung zu kommen, muß man also diesem Modell die folgenden Forderungen stellen:

a) Das Verhältnis des magnetischen Momentes des Elektrons zum mechanischen muß für die Eigenrotation doppelt so groß sein als für die Umlaufbewegung²⁾.

b) Die verschiedenen Orientierungen vom R zur Bahnebene (oder K) des Elektrons muß, vielleicht in Zusammenhang mit einer HEISENBERG-WENTZELschen Mittelungsvorschrift³⁾, die Erklärung des Relativitätsdoubletts liefern können.

G. E. UHLENBECK und S. GOUDSMIT.

Leiden, den 17. Oktober 1925.

Instituut voor Theoretische Natuurkunde.

FIFTY YEARS OF SPIN

Personal reminiscences

How one student who was undecided whether to pursue a career in physics or history and another who had not taken his mechanics exam identify the fourth atomic quantum number with a rotation of the electron.

George E. Uhlenbeck

FIFTY YEARS OF SPIN

It might as well be spin

Samuel A. Goudsmit

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NATURE

[APRIL 10, 1926]

Letters to the Editor.

[The Editor does not hold himself responsible for opinions expressed by his correspondents. Neither can he undertake to return, nor to correspond with the writers of, rejected manuscripts intended for this or any other part of NATURE. No notice is taken of anonymous communications.]

The Motion of the Spinning Electron.

In a letter published in NATURE of February 20, p. 264, Messrs. Uhlenbeck and Goudsmit have shown how great difficulties which atomic theory had met in the attempt to explain spectral structure and Zeeman effects, can be avoided by using the idea of the spinning electron. Although their theory is in complete qualitative agreement with observation, it involved an apparent quantitative discrepancy. The value of the precession of the spin axis in an external magnetic field required to account for Zeeman effects seemed to lead to doublet separations twice those which are observed. This discrepancy, however, disappears when the kinematical problem concerned is examined more closely from the point of view of the theory of relativity.

As usual, letters in heavy type will denote vectors. The anomalous Zeeman effect seems to require that the spin axis of the electron precesses about an external magnetic field H with angular velocity

$$\frac{e}{mc} H, \quad (A)$$

where c is the velocity of light and $-e, m$ are the electronic charge and mass. Suppose such a spinning electron moves with velocity v through electric field E . At first sight it would seem that, being subject to magnetic field

$$H = \frac{1}{c} [E \times v],$$

the spin axis will precess about the instantaneous normal to the orbital plane with angular velocity

$$\frac{e}{mc^2} [E \times v]. \quad (B)$$

As the mean value of this expression is just twice the angular velocity with which the perihelion of the orbit rotates on account of the variation of mass of the electron, this would lead to twice the observed doublet separation.

There is, however, an error in the above reasoning; the precession of the spin axis so calculated is its precession in a system of co-ordinates (2) in which the centre of the electron is momentarily at rest. System (2) is obtained from system (1), in which the electron is moving and the nucleus at rest, by a Lorentz transformation with velocity v . If the acceleration of the electron is f , and system (3) is obtained from system (1) by a Lorentz transformation with velocity $v + fdt$, then the precession which an observer at rest with respect to the nucleus would observe, and which should be summed to give the secular precession, is that precession which would turn the direction of the spin axis at time t in (2) into its direction at time $t + dt$ in (3) if both directions were regarded as directions in (1). To a first approximation system (3) is obtained from system (2) by a Lorentz transformation with velocity fdt together with a rotation $(1/2c^2)[v \times fdt]$. Thus the observed rate of precession will be, to a first approximation,

$$\frac{e}{mc^2} [E \times v] - \frac{1}{2c^2} [v \times f].$$

NO. 2945, VOL. 117]

To a first approximation

$$f = -\frac{e}{m} E,$$

so the rate of precession is

$$\frac{e}{2mc^2} [E \times v], \quad (C)$$

just half the expression (B).

The interpretation of the fine structure of the hydrogen lines proposed by Messrs. Uhlenbeck and Goudsmit now no longer involves any discrepancy. In fact, as Dr. Pauli and Dr. Heisenberg have kindly communicated in letters to Prof. Bohr, it seems possible to treat the doublet separation as well as the anomalous Zeeman effect rigorously on the basis of the new quantum mechanics. The result seems to be full agreement with experiment when the calculation is based on formulæ (A) and (C).

I hope in a later paper to develop the above kinematical argument in greater detail.

In conclusion, I wish to express my appreciation of the encouragement and help of Prof. Bohr and Dr. Kramers.

L. H. THOMAS.
Universitets Institut for Teoretisk Fysik,
Copenhagen, February 20.

Genes and Linkage Groups in Genetics.

Now that Prof. MacBride has delivered himself of his final reply to Prof. Huxley on the subject of linkage and genes, I should like to offer a few comments on a point from which further confusion might easily arise in the future. It concerns the use of the word 'linkage.' This term was originally proposed by Prof. Morgan, and the first evidence of its appear-



Llewellyn Thomas

done, the case, interesting as it may be in other

► The letter in *Sonderdruck aus Die Naturwissenschaften* is dated 17 Oct 1925.

1928 and later: Dirac and Wigner

Dirac reverse-engineered the relativistic energy equation:

$$E^2 = m^2 c^4 + c^2 \mathbf{p}^2 \quad \Rightarrow \quad \hat{H}^2 = m^2 c^4 + c^2 \hat{\mathbf{p}}^2$$

Assume:

$$\hat{H} = \alpha_0 m c^2 + c (\alpha_1 \hat{p}_1 + \alpha_2 \hat{p}_2 + \alpha_3 \hat{p}_3)$$

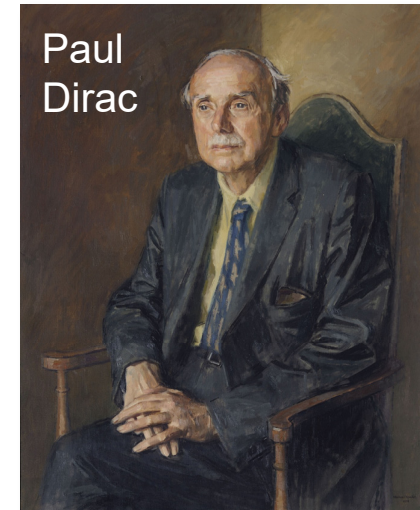
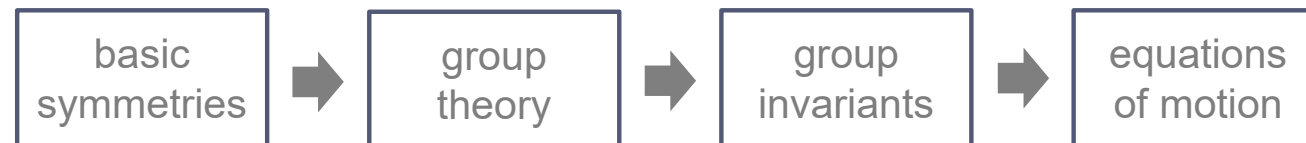
No scalar solutions, but it turns out α_k can be 4×4 matrices:

$$\alpha_0 = \begin{pmatrix} +1 & \mathbf{0} \\ \mathbf{0} & -1 \end{pmatrix}, \quad \alpha_n = \begin{pmatrix} \mathbf{0} & \boldsymbol{\sigma}_n \\ \boldsymbol{\sigma}_n & \mathbf{0} \end{pmatrix}, \quad n \in \{1, 2, 3\}$$

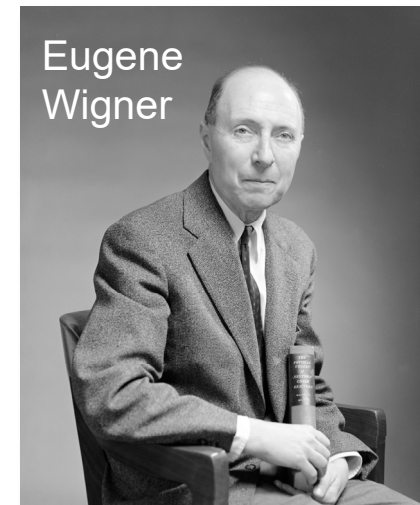
unit Pauli matrices

This dubious hack (1928) became known as *Dirac's coup*.

A rigorous framework was published in 1931 by Wigner:



Paul Dirac



Eugene Wigner

► Wigner type Lie-algebraic arguments are now standard *modus operandi* in theoretical physics.

Causality and continuity of time

Assume reality is knowable:

Ψ ← descriptor (not necessarily wavefunction)

Assume causal determinism:

$$\Psi(t+t') = \hat{U}(t, t') \Psi(t)$$

Assume time is uniform:

$$\hat{U}(t, t') = \hat{U}(\tau), \quad \tau = t' - t$$

$$\Psi(t+\tau) = \hat{U}(\tau) \Psi(t)$$

Assume reality is continuous:

$$\Psi(t+\tau) =$$

$$= \left[\sum_{n=0}^{\infty} \frac{\tau^n}{n!} \frac{\partial^n}{\partial t^n} \right] \Psi(t) \Rightarrow \hat{U}(\tau) = \exp \left[\tau \frac{\partial}{\partial t} \right]$$

time propagator
(a group)

time
evolution
generator

Eigensystem of the generator:

$$\frac{\partial}{\partial t} [e^{-i\omega t}] = -i\omega [e^{-i\omega t}] \leftarrow \text{irreducible representations}$$

Arbitrary representation:

$$\frac{\partial}{\partial t} = -i \begin{bmatrix} \omega_1 & & \\ & \omega_2 & \\ & & \dots \end{bmatrix} \Rightarrow \boxed{\frac{\partial}{\partial t} = -i\mathbf{H}}$$

► In any knowable and continuous reality with unchanging rules, the EOM must have this form.

Physical symmetries: translations

Generators and group elements are obtained in the same way:

coefficient needed to match units $\longrightarrow$

$$\hat{p}_0 = i \frac{\partial}{\partial ct} \quad \hat{p}_1 = -i \frac{\partial}{\partial x} \quad \hat{p}_2 = -i \frac{\partial}{\partial y} \quad \hat{p}_3 = -i \frac{\partial}{\partial z}$$

$$\hat{T}(\mathbf{r}) = \exp \left[-i \left(\hat{p}_0 r_0 + \hat{p}_1 r_1 + \hat{p}_2 r_2 + \hat{p}_3 r_3 \right) \right], \quad \mathbf{r} = [ct \quad x \quad y \quad z]$$

Casimir operator:

$$\hat{m}^2 = \hat{p}_0^2 - \hat{p}_1^2 - \hat{p}_2^2 - \hat{p}_3^2$$

has to be mass $\nearrow$ energy squared $\nearrow$ momentum squared

After fixing units of mass:

$$E^2 = m^2 c^4 + c^2 \mathbf{p}^2$$

relativistic energy-momentum relation

In the massless case:

$$\hat{p}_0^2 = \hat{p}_1^2 + \hat{p}_2^2 + \hat{p}_3^2$$

$\Downarrow$

$$\frac{\partial^2}{\partial t^2} = c^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)$$

wave equation $\nwarrow$

$\nearrow$ propagation speed for massless waves, has to be speed of light

Energy = linear momentum in time.

► Consequences: conservation of rest mass and linear momentum; speed of light.

Physical symmetries: rotations

Rotations are a continuous group:

$$\exp\left[\alpha \frac{\partial}{\partial \varphi}\right] f(\varphi) = f(\varphi + \alpha)$$

↓

$$\mathbf{R}(\mathbf{n}, \alpha) = \exp\left[-\alpha \frac{\partial}{\partial \varphi}\right]$$

axis
angle

Casimir operator:

$$\hat{L}^2 = \hat{L}_X^2 + \hat{L}_Y^2 + \hat{L}_Z^2$$

- commutes with all group elements
- enumerates irreps
- corresponds to a conserved quantity

$$\begin{cases} \mathbf{R}(\varphi_1) \mathbf{R}(\varphi_2) = \mathbf{R}(\varphi_1 + \varphi_2) & \text{closure} \\ \mathbf{1} = \mathbf{R}(0) & \text{unit element} \\ \mathbf{R}(\varphi) \mathbf{R}(-\varphi) = \mathbf{1} & \text{inverse element} \end{cases}$$

Cartesian rotation generators:

$$\text{XY plane: } -i \frac{\partial}{\partial \varphi} = -i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right) = \hat{L}_Z$$

$$\left[\hat{L}_X, \hat{L}_Y \right] = i \hat{L}_Z, \quad \left[\hat{L}_Z, \hat{L}_X \right] = i \hat{L}_Y, \quad \left[\hat{L}_Y, \hat{L}_Z \right] = i \hat{L}_X$$

structure relations

group element

algebra

$$\mathbf{R}(\mathbf{n}, \alpha) = \exp\left[-i \left(n_X \hat{L}_X + n_Y \hat{L}_Y + n_Z \hat{L}_Z \right) \alpha\right]$$

group parameters

exponential map

generators

► Orbital angular momentum is conserved because reality is isotropic.

Michelson-Morley experiment (1887)

Speed of light was the same in all reference frames. Lorentz suggested:

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \mathbf{K}_x(v) \begin{bmatrix} ct \\ x \end{bmatrix}$$

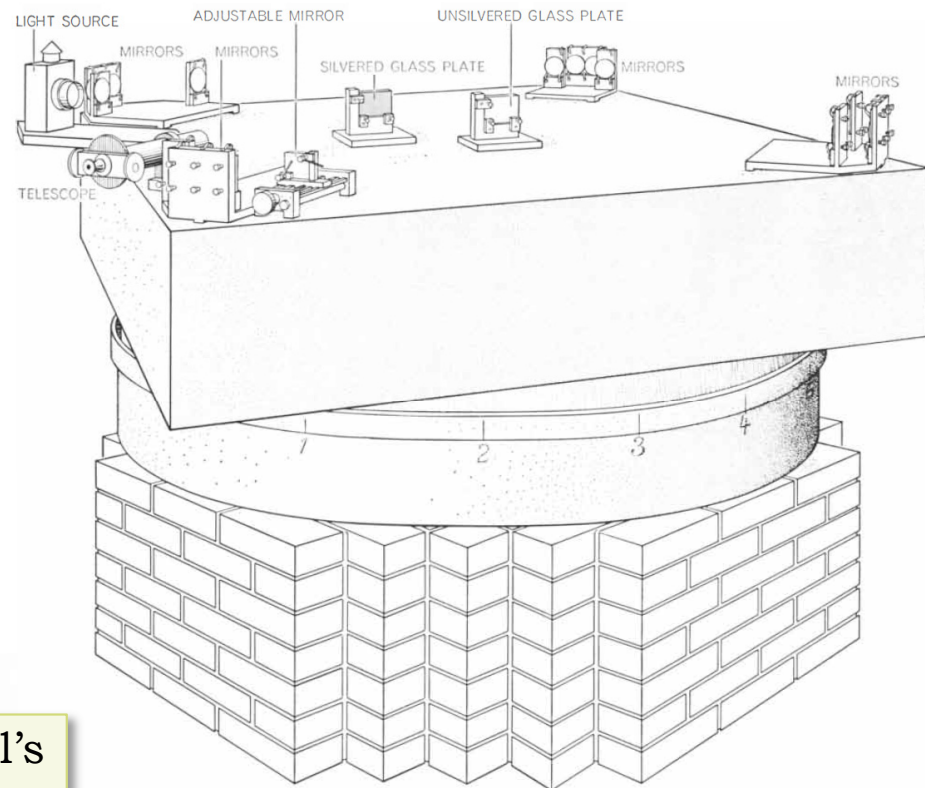
- space to be shrunk
- time to be dilated
- must be a group

The only transform that fits the bill:

$$\mathbf{K}_x(v) = \frac{1}{\sqrt{1-v^2/c^2}} \begin{bmatrix} 1 & -v/c \\ -v/c & 1 \end{bmatrix}$$

Lorentz had already seen this for Maxwell's equations – but the target of the transform this time was... experimental apparatus!

VERSUCH EINER THEORIE DER ELECTRISCHEN UND
OPTISCHEN ERSCHEINUNGEN IN BEWEGTEN
KÖRPERN



Physical symmetries: Lorentz group

Consider the group of all rotations and boosts. Cosmetic transformations:

$$\begin{bmatrix} ct' \\ x' \end{bmatrix} = \mathbf{K}_x(v) \begin{bmatrix} ct \\ x \end{bmatrix}, \quad \mathbf{K}_x(v) = \frac{1}{\sqrt{1-v^2/c^2}} \begin{bmatrix} 1 & -v/c \\ -v/c & 1 \end{bmatrix}$$

$$v/c = \tanh \varphi \quad \Rightarrow \quad \mathbf{K}_x(\varphi) = \begin{pmatrix} \cosh(\varphi) & -\sinh(\varphi) \\ -\sinh(\varphi) & \cosh(\varphi) \end{pmatrix} = \exp \left[-i \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \varphi \right]$$

“rapidity”
↓

After some math, generators are hyperbolic “stretch and squeeze” operators:

$$\hat{K}_x = -i \left(x \frac{\partial}{\partial ct} + ct \frac{\partial}{\partial x} \right) \quad \hat{K}_y = -i \left(y \frac{\partial}{\partial ct} + ct \frac{\partial}{\partial y} \right) \quad \hat{K}_z = -i \left(z \frac{\partial}{\partial ct} + ct \frac{\partial}{\partial z} \right)$$

Inspection of structure relations gives the first glimpse of a new property:

$$\left[\hat{L}_n, \hat{L}_k \right] = i\epsilon_{nkm} \hat{L}_m \quad \left[\hat{K}_n, \hat{K}_k \right] = -i\epsilon_{nkm} \hat{L}_m \quad \left[\hat{L}_n, \hat{K}_k \right] = i\epsilon_{nkm} \hat{K}_m$$

the usual rotations

new rotations

► In Minkowski space-time, a rotation may also be accomplished by a superposition of boosts.

Intermediate summary: it's relativity!

Euclidean space

$$r^2 = x^2 + y^2 + z^2 \quad \leftarrow \text{Euclidean distance}$$

angular momentum operators

$$\hat{L}_Z = -i \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x} \right), \text{ etc.}$$

$$\hat{R}(\mathbf{n}, \varphi) = \exp \left[-i \underbrace{(n_x \hat{L}_x + n_y \hat{L}_y + n_z \hat{L}_z)}_{\text{direction}} \varphi \right]$$

rotation angle



$$\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2$$

Casimir invariant
(a conserved quantity)

"every continuous symmetry leads to a conservation law"

Standard commutation relations:

$$[\hat{L}_x, \hat{L}_y] = i\hat{L}_z, \text{ etc.}$$

cannot rotate a point mass
(infinite moment of inertia)

Minkowski space-time

$$r^2 = x^2 + y^2 + z^2 - c^2 t^2 \quad \leftarrow \text{Minkowski distance}$$

Lorentz boost operators

$$\hat{K}_Z = -i \left(z \frac{\partial}{\partial ct} + ct \frac{\partial}{\partial z} \right), \text{ etc.}$$

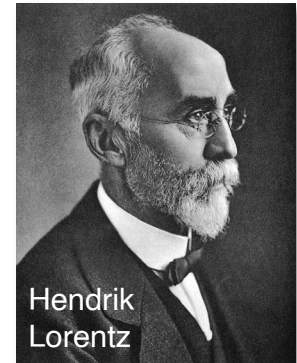
$$[\hat{K}_x, \hat{K}_y] = -i\hat{L}_z, \text{ etc.}$$

The commutator of two Lorentz boosts is a rotation!

angular momentum something else

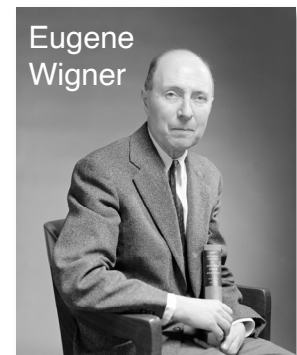
$$\hat{C}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2 - \hat{K}_x^2 - \hat{K}_y^2 - \hat{K}_z^2$$

updated Casimir invariant
(a different conserved quantity)



A new symmetry:

- space is shrunk
- time is dilated



demand conservation and rearrange



Dirac's equation

► There are more ways of rotating things in Minkowski space-time than there were in Euclidean 3D.

Irreducible representations of Lorentz group

There are two Casimir operators:

Eigenfunctions are horrendous, but Wigner found a workaround:

$$\begin{cases} \hat{M}_n = (\hat{L}_n + i\hat{K}_n)/2 \\ \hat{N}_n = (\hat{L}_n - i\hat{K}_n)/2 \end{cases} \quad n \in \{X, Y, Z\}$$

Commutation relations simplify:

$$\begin{aligned} [\hat{M}_n, \hat{M}_k] &= i\epsilon_{nkm} \hat{M}_m \\ [\hat{N}_n, \hat{N}_k] &= i\epsilon_{nkm} \hat{N}_m \\ [\hat{M}_n, \hat{N}_k] &= 0 \end{aligned}$$

we have seen these before

Lorentz group envelope period:

“total” angular momentum

$$\begin{aligned} \hat{C}_1 &= \hat{L}_X^2 + \hat{L}_Y^2 + \hat{L}_Z^2 - \hat{K}_X^2 - \hat{K}_Y^2 - \hat{K}_Z^2 \\ \hat{C}_2 &= \hat{L}_X \hat{K}_X + \hat{L}_Y \hat{K}_Y + \hat{L}_Z \hat{K}_Z \end{aligned}$$

Two groups in mathematics fit the bill:

SU(2)

$$[\mathbf{S}_i, \mathbf{S}_j] = i\epsilon_{ijk} \mathbf{S}_k$$

$$\exp[-2\pi i \mathbf{S}_Y] = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\exp[-4\pi i \mathbf{S}_Y] = \begin{pmatrix} +1 & 0 \\ 0 & +1 \end{pmatrix}$$

Group period:

4π

2π

SO(3)

$$[\mathbf{L}_i, \mathbf{L}_j] = i\epsilon_{ijk} \mathbf{L}_k$$

$$\exp[-2\pi i \mathbf{L}_Y] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\exp[-4\pi i \mathbf{L}_Y] = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\exp\left[-i \frac{\hat{L}_X + i\hat{K}_X}{2} \alpha\right] = \exp\left[-i\hat{L}_X \frac{\alpha}{2}\right] \exp\left[\hat{K}_X \frac{\alpha}{2}\right] \quad 4\pi$$

► There are two instances of SU(2) in Lorentz group. Reason: the presence of boosts.

Irreducible representations of Lorentz group

The irreps of Lorentz group can now be built from the irreps of SU(2):

$$\mathfrak{so}(3,1)_{\mathbb{C}} \cong \mathfrak{su}(2)_{\mathbb{C}} \oplus \mathfrak{su}(2)_{\mathbb{C}}$$

↑
Lorentz algebra
↙ ↘
irreps are known for these

$$\begin{cases} \mathbf{M}_k^{(j_1, j_2)} = \mathbf{S}_k^{(j_1)} \otimes \mathbf{1}^{(j_2)} \\ \mathbf{N}_k^{(j_1, j_2)} = \mathbf{1}^{(j_1)} \otimes \mathbf{S}_k^{(j_2)} \end{cases} \Leftrightarrow (j_1, j_2)$$

Inversion exchanges **M** and **N**. This knocks out C_2 and constrains the irreps:

$$\hat{C}_1 = \hat{L}^2 - \hat{K}^2 \quad (j, j) \quad \text{or} \quad (j_1, j_2) \oplus (j_2, j_1) \quad \leftarrow \text{these correspond to different types of particles and fields}$$

For electrons, only one irrep matches experimental observations:

$$\left(\frac{1}{2}, 0\right) \oplus \left(0, \frac{1}{2}\right):$$

$$\mathbf{L}_n = \begin{pmatrix} \boldsymbol{\sigma}_n/2 & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\sigma}_n/2 \end{pmatrix}, \quad \mathbf{K}_n = \begin{pmatrix} i\boldsymbol{\sigma}_n/2 & \mathbf{0} \\ \mathbf{0} & -i\boldsymbol{\sigma}_n/2 \end{pmatrix}$$

$$\boldsymbol{\sigma}_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \boldsymbol{\sigma}_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad \leftarrow \text{unit Pauli matrices}$$

► ...but Lorentz group has no time translation generator that is needed to get an equation of motion.

Physical symmetries: Poincare group

Put all momenta, rotations, and boosts into one group:

$$\begin{aligned}
 [\hat{L}_n, \hat{L}_k] &= i\epsilon_{nkm} \hat{L}_m & [\hat{K}_n, \hat{K}_k] &= -i\epsilon_{nkm} \hat{L}_m & [\hat{L}_n, \hat{K}_k] &= i\epsilon_{nkm} \hat{K}_m \\
 [\hat{L}_n, \hat{p}_k] &= i\epsilon_{nkm} \hat{p}_m & [\hat{L}_n, \hat{p}_0] &= 0 & [\hat{p}_\mu, \hat{p}_\nu] &= 0 \\
 [\hat{K}_n, \hat{p}_k] &= i\delta_{nk} \hat{p}_0 & [\hat{K}_n, \hat{p}_0] &= i\hat{p}_n \\
 n, k, m &\in \{1, 2, 3\} & \mu, \nu &\in \{0, 1, 2, 3\}
 \end{aligned}$$

$\hat{C}_M = \hat{p}^2 - \hat{p}_0^2$ ← mass
 $\hat{C}_L = \hat{L}^2 - \hat{K}^2$ ← from Lorentz group
2nd time derivative

Multiplying up Casimir operators for the $[\frac{1}{2}, 0](+)[0, \frac{1}{2}]$ irrep:

$$\begin{cases} \hat{p}^2 = \hat{p}_0^2 - m^2 c^2 \\ \frac{\mathbf{L}^2 - \mathbf{K}^2}{2} = \frac{3}{4} \end{cases} \quad (\mathbf{L}^2 - \mathbf{K}^2) \hat{p}^2 = \frac{3}{2} (\hat{p}_0^2 - m^2 c^2) \quad \Rightarrow \quad \begin{pmatrix} \boldsymbol{\sigma}^2 \hat{p}^2 & 0 \\ 0 & \boldsymbol{\sigma}^2 \hat{p}^2 \end{pmatrix} = \hat{p}_0^2 - m^2 c^2$$

Unit Pauli matrices have the following property:

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b}) \quad \Rightarrow \quad \boldsymbol{\sigma}^2 \hat{p}^2 = (\boldsymbol{\sigma} \cdot \mathbf{p})^2$$

► Poincare group classifies systems by only two properties: mass and total momentum.

Dirac equation

After the Pauli matrix relation is used, we obtain:

$$\hat{p}_0^2 = \begin{pmatrix} m^2 c^2 + (\boldsymbol{\sigma} \cdot \mathbf{p})^2 & 0 \\ 0 & m^2 c^2 + (\boldsymbol{\sigma} \cdot \mathbf{p})^2 \end{pmatrix}$$

After inserting the time translation generator:

$$\frac{\partial^2}{\partial t^2} = - \begin{bmatrix} m^2 c^4 + c^2 (\boldsymbol{\sigma} \cdot \mathbf{p})^2 & 0 \\ 0 & m^2 c^4 + c^2 (\boldsymbol{\sigma} \cdot \mathbf{p})^2 \end{bmatrix}$$

Adding electromagnetic potentials:

$$\mathbf{p} \rightarrow \boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}, \quad i\partial_t \rightarrow i\partial_t - q\varphi$$

After picking the positive time direction:

$$\frac{\partial}{\partial t} = -i \begin{bmatrix} q\varphi + mc^2 & c(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) & q\varphi - mc^2 \end{bmatrix} \quad \leftarrow \text{Dirac equation}$$

Problem:

the equation of motion *must* be first order in time!

$$\frac{\partial}{\partial t} = -i\mathbf{H}$$

Matrix square root is a well defined operation:

$$\frac{\partial}{\partial t} = \pm i \begin{bmatrix} +mc^2 & c(\boldsymbol{\sigma} \cdot \mathbf{p}) \\ c(\boldsymbol{\sigma} \cdot \mathbf{p}) & -mc^2 \end{bmatrix}$$



► Dirac essentially guessed this equation, the derivation above is only possible in hindsight.

Positive energy subspace

For a free electron in the absence of electric fields:

$$\mathbf{H} = \begin{bmatrix} mc^2 & c(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) & -mc^2 \end{bmatrix} = \underbrace{\begin{bmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{bmatrix}}_{\text{enormous splitting } (\sim \text{MeV})} + \underbrace{\begin{bmatrix} 0 & c(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) & 0 \end{bmatrix}}_{\text{some kind of interaction } (\sim \text{eV})}$$

$\boldsymbol{\pi} = \mathbf{p} - q\mathbf{A}$
 $\hbar\omega = 2mc^2$
 zitterbewegung (zeptoseconds)

negative energy states

Interaction representation:

$$\mathbf{H}_0 = \begin{bmatrix} mc^2 & 0 \\ 0 & -mc^2 \end{bmatrix}$$

$$\mathbf{H}_1 = \begin{bmatrix} 0 & c(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) \\ c(\boldsymbol{\sigma} \cdot \boldsymbol{\pi}) & 0 \end{bmatrix}$$

$$\mathbf{H}_1^{\text{R}}(t) = e^{+i\mathbf{H}_0 t} \mathbf{H}_1 e^{-i\mathbf{H}_0 t}$$

Average Hamiltonian theory, zeroth order:

$$\frac{1}{T} \int_0^T \mathbf{H}_1^{\text{R}}(t) dt = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad T = \frac{2\pi}{\omega}, \quad \omega = 2mc^2$$

Average Hamiltonian theory, first order:

$$-\frac{i}{2T} \int_0^T \int_0^{t_1} [\mathbf{H}_1^{\text{R}}(t_1), \mathbf{H}_1^{\text{R}}(t_2)] dt_2 dt_1 = \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2}{2m} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

► First order average Hamiltonian stays in the positive energy subspace!

Pauli Hamiltonian

To first order in the average Hamiltonian theory:

$$\mathbf{H} = \frac{(\boldsymbol{\sigma} \cdot \boldsymbol{\pi})^2}{2m} = \frac{\boldsymbol{\pi} \cdot \boldsymbol{\pi} + i\boldsymbol{\sigma} \cdot (\boldsymbol{\pi} \times \boldsymbol{\pi})}{2m} = \frac{\pi^2}{2m} + i \frac{\boldsymbol{\sigma} \cdot (\boldsymbol{\pi} \times \boldsymbol{\pi})}{2m}$$

usual kinetic energy something else

For unit Pauli matrices:

$$(\boldsymbol{\sigma} \cdot \mathbf{a})(\boldsymbol{\sigma} \cdot \mathbf{b}) = \mathbf{a} \cdot \mathbf{b} + i\boldsymbol{\sigma} \cdot (\mathbf{a} \times \mathbf{b})$$

Picking apart the vector cross product:

$$\begin{aligned} \boldsymbol{\pi} \times \boldsymbol{\pi} &= \mathbf{p} \times \mathbf{p} - q(\mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p}) + q^2 \mathbf{A} \times \mathbf{A} = \\ &= -q(\mathbf{p} \times \mathbf{A} + \mathbf{A} \times \mathbf{p}) = \dots = i\hbar q \mathbf{B} \end{aligned}$$

zero zero long story

Reminders:

$$\begin{aligned} \mathbf{E} &= -\nabla \phi & \mathbf{p} &= -i\nabla \\ \mathbf{B} &= \nabla \times \mathbf{A} \end{aligned}$$

Further corrections are:

$$\mathbf{H}_{\text{so}} = -\frac{e\hbar}{4m^2c^2} \boldsymbol{\sigma} \cdot [(-\nabla \phi) \times \mathbf{p}]$$

$$\mathbf{H}_{\text{D}} = +\frac{e\hbar^2}{8m^2c^2} \nabla^2 \phi$$

$$\mathbf{H}_{\text{RM}} = -\frac{e\hbar}{8m^3c^2} \boldsymbol{\pi}^4$$

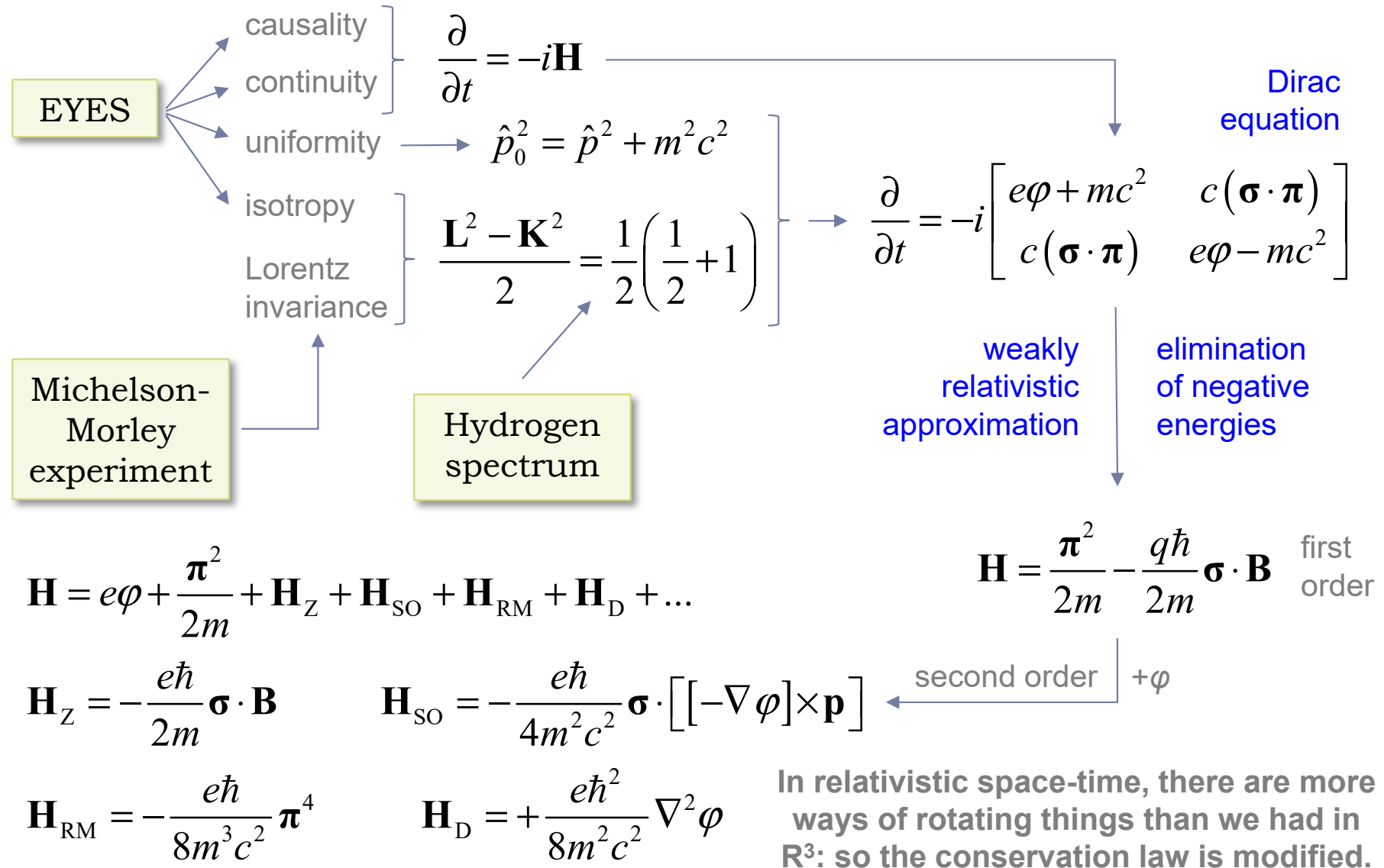
Final result to first order in AHT:

$$\mathbf{H} = \frac{\pi^2}{2m} - \frac{q\hbar}{2m} \boldsymbol{\sigma} \cdot \mathbf{B}$$

kinetic energy Zeeman interaction

► The quantity corresponding to $\boldsymbol{\sigma}$ gives electrons a magnetic moment.

Overview of the path we took



► Spin makes an appearance because time is a degree of freedom in special relativity.