Week 7 workshop exercises

1. Calculate the first four terms of the Taylor series around $x_0 = 0$ for the following functions:

   (a) $\dfrac{1}{1-3x}$ (b) $\dfrac{1}{1+5x^2}$ (think first) (c) $\dfrac{1}{2+x}$

   Try to guess the infinite sum form of the series, and then use the ratio test to find the values of x for which the series converge.

2. Find the linear high-temperature approximation for the Fermi-Dirac distribution that describes the average number of fermions n in a single-particle state with the energy U in a system with the total chemical potential μ:

$$n(T) = \frac{1}{e^{(U-\mu)/kT} + 1}$$

   Note: make a substitution $x = (U-\mu)/kT$, then get the Taylor series up to the term with the first power of x around $x = 0$, then reverse the substitution.

3. Using the ratio test, find the radius of convergence for each of the following series:

   (a) $\displaystyle\sum_{m=0}^{\infty} \frac{x^m}{4^m}$ (b) $\displaystyle\sum_{r=0}^{\infty} (-1)^r x^{2r}$ (c) $\displaystyle\sum_{n=1}^{\infty} n x^n$ (d) $\displaystyle\sum_{n=1}^{\infty} \frac{x^n}{n^2}$

4. The following are the Taylor series for sine and cosine:

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} \qquad \cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

   (a) write out the first four terms of each series explicitly;

   (b) differentiate the explicit expression for the sine series and demonstrate that the derivative is equal to the cosine series;

   (c) integrate the explicit expression for the sine series and demonstrate that the integral is equal to the negative of the cosine series plus a constant.

5. Find the Lagrange interpolant for the function taking the following values: $f(0) = 0$, $f(1) = 1$, $f(2) = 1$, $f(3) = 2$. Calculate the value of the interpolant at $x = 3/2$.