① 

a) $\vec{x} = \begin{pmatrix} 1 \\ 4 \\ 0 \\ 2 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 2 \\ -2 \\ 1 \\ 3 \end{pmatrix}$

$$|\vec{x}| = \sqrt{1+16+0+4} = \sqrt{21}$$

$$|\vec{y}| = \sqrt{4+4+1+9} = \sqrt{18}$$

$$\langle x|y\rangle = 1\cdot 2 - 4\cdot 2 + 0\cdot 1 + 2\cdot 3 = 0$$

b) $\vec{x} = \begin{pmatrix} 0 \\ i \\ 0 \\ -i \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 0 \\ 1 \\ 1 \\ 1 \end{pmatrix}$

$$\langle x|y\rangle = \sum_k x_k^* y_k =$$

$$|\vec{x}| = \sqrt{0+1+0+1} = \sqrt{2}$$

$$= 0\cdot 0 + (-i)\cdot 1 + 0\cdot 1 - (-i)\cdot 1 =$$

$$|\vec{y}| = \sqrt{0+1+1+1} = \sqrt{3}$$

$$= -i + i = 0.$$

c) $\vec{x} = \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix}$ $\vec{y} = \begin{pmatrix} 0 \\ i \\ 0 \\ i \end{pmatrix}$

$$|\vec{x}| = \sqrt{0+1+0+1} = \sqrt{2}$$

$$|\vec{y}| = \sqrt{0+1+0+1} = \sqrt{2}$$

$$\langle x|y\rangle = 1\cdot 0 + 0\cdot i + 1\cdot 0 + 0\cdot i = 0.$$

② No, because the basis of a two-dimensional space has two vectors. Any other vector is by definition their linear combination. Therefore any set of three 2D vectors is linearly dependent.

(3) All properties are satisfied. Because any $3\times3$ matrix has 9 independent elements, the space is 9-dimensional. Other possibilities are functions (infinite-dimensional), differential operators (infinite-dimensional), and bank accounts (one-dimensional).

(4) None of the four vectors is a linear combination of other three.

(5) By direct inspection:

$$\begin{pmatrix} 3 \\ 7i \\ 8 \\ 0 \end{pmatrix} = 3\vec{X}_1 + 7i\vec{X}_2 + 8\vec{X}_3 \ .$$

(6)

$$\vec{X}_1 = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad \vec{X}_2 = \begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$\vec{y} = \begin{pmatrix} i \\ 0 \end{pmatrix} \qquad \vec{y} = \alpha_1 \vec{X}_1 + \alpha_2 \vec{X}_2$$

$$\begin{cases} \langle x_1 | y \rangle = \alpha_1 \langle x_1 | x_1 \rangle + \alpha_2 \langle x_1 | x_2 \rangle \\ \langle x_2 | y \rangle = \alpha_1 \langle x_2 | x_1 \rangle + \alpha_2 \langle x_2 | x_2 \rangle \end{cases} \Rightarrow$$

$$\Rightarrow \begin{cases} i = \alpha_1 \cdot 2 + \alpha_2 (1+i) \\ i = \alpha_1 (-i) + \alpha_2 \cdot 2 \end{cases} \Rightarrow \begin{cases} \alpha_1 = \frac{1+i}{2} \\ \alpha_2 = \frac{-1+i}{2} \end{cases}$$

(7) By direct inspection, the orthogonal pairs are: $\vec{X}_1$ and $\vec{X}_3$, $\vec{X}_2$ and $\vec{X}_3$.

(8) Consider an unknown vector $[a\ b\ c]^T = \vec{x}_3$

We must have:

$$\begin{cases} \langle \vec{x}_1 | \vec{x}_3 \rangle = 0 \\ \langle \vec{x}_2 | \vec{x}_3 \rangle = 0 \end{cases} \Rightarrow \begin{cases} a + b + c = 0 \\ a - b = 0 \end{cases}$$

Not enough equations, but that is because the vector can have any length. Let's pick $a = 1$. Then $b = 1$, $c = -2$.

$$\vec{x}_3 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \qquad |\vec{x}_3| = \sqrt{1+1+4} = \sqrt{6}.$$

And therefore the orthonormal basis is:

$$\left\{ \frac{1}{\sqrt{3}}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}, \frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \right\}.$$