Week 26 problem set

1. Demonstrate that $Y_{2,0}(\theta,\varphi)$ given in Table 1 of Lecture 1:

   (a) satisfies Equation 18 from the same lecture,

   (b) satisfies $\hat{L}_Z Y_{lm}(\theta,\varphi) = m Y_{lm}(\theta,\varphi)$ where $\hat{L}_Z = -i\,\partial/\partial\varphi$.

2. Demonstrate that $Y_{2,1}(\theta,\varphi)$ and $Y_{1,0}(\theta,\varphi)$ given in Table 1 of Lecture 1 are mutually orthogonal and normalised with respect to the following norm and scalar product:

$$\langle Y_{ab}(\theta,\varphi) | Y_{cd}(\theta,\varphi) \rangle = \int_0^{2\pi}\int_0^{\pi} \left[Y_{ab}^*(\theta,\varphi) Y_{cd}(\theta,\varphi) \sin\theta \right] d\theta d\varphi$$

$$\| Y_{ab}(\theta,\varphi) \| = \sqrt{\langle Y_{ab}(\theta,\varphi) | Y_{ab}(\theta,\varphi) \rangle}$$

3. Try to prove that the eigenvectors of Hermitian matrices ($\mathbf{A}^\dagger = \mathbf{A}$ where dagger stands for conjugate-transpose) corresponding to different eigenvalues are orthogonal.

4. The probability of finding the electron somewhere inside a shell of radius r and of thickness dr is proportional to $r^2 |R_{nl}(r)|^2 dr$. The factor $|R_{nl}(r)|^2$ gives the probability density at radius r, and the factor $r^2 dr$ is proportional to the volume of the shell. Prove the statement made in Equation 8 of Lecture 2 by following through the optimisation procedure.

5. Calculate the expectation value $\langle r \rangle = \langle \psi | r | \psi \rangle$ for the ground state of hydrogen. Do not forget the angular integral and the spherical coordinate Jacobian.

6. Calculate the wavelength of the transition between the $n=1$ and $n=2$ states of hydrogen (this is called the *Lyman-α line*).

7. Calculate the wavelength of the transition between the $n=2$ and $n=3$ states of hydrogen (this is called the *Balmer-α line*).

8. The energy of a *hydrogen-like atom* (one electron, but a different nuclear charge) is given by

$$E_n = -\frac{\mu e^4 Z^2}{2(4\pi\varepsilon_0)^2 \hbar^2 n^2}$$

   where Z is an integer giving the nuclear charge in units of proton charge. Calculate the wavelength of the Balmer-α line in the He^+ ion. Remember to update the reduced mass.

9. The energy levels of a one-electron atom are proportional to the reduced mass of the system, and so there will be a small energy difference between the levels of hydrogen and deuterium. As a result, the spectral lines of deuterium will be slightly shifted in frequency relative to those of hydrogen. This is known as *isotope shift*. Calculate the isotope shift in frequency units for the Lyman-α line of hydrogen and deuterium.

10. Positronium is a bound system of an electron and a positron. What is the wavelength of the Balmer-α transition in positronium?

11. Muonic hydrogen is a bound system of a muon orbiting a proton. Muons are heavier than electrons. How would this influence the size of the resulting "atom" in its ground state?